MATH 3322
Supplemental notes on Cartesian products in algebra

Recall: Cartesian product of sets.

Let $(A_j)_{j\in J}$ be a *family of sets*. By that we mean that we have a function f with domain J, and for each $j \in J$, $f(j)$ is some set which we label as A_j. So the notation for a family of sets allows us to talk about the *range* of the function f without ever having to name f explicitly.

The *cartesian product* of the family, $\prod_{j\in J} A_j$, is the set of all functions $p : I \to \bigcup_{j\in J}$ such that for all $j \in J$, $p(j) \in A_j$. It is a (moderately straightforward) exercise to show that if J is a finite set, then the product is well-defined, but the existence of the product for all infinite families of non-empty sets is exactly the Axiom of Choice. If $J = \{0, 1, \ldots, n-1\}$ then the cartesian product is just what we usually think of as the set of all n-tuples $\langle a_0, \ldots, a_{n-1} \rangle$ with $a_j \in A_j$.

In imitation of this, instead of using function notation to indicate elements of the cartesian product, we use "vector" notation and write things like $\overline{a}$, $\mathbf{a}$, or $\vec{a}$ for elements of the cartesian product, with $\overline{a} = (a_j)_{j\in J}$. We call a_j the *j-th coordinate* of $\overline{a}$. Again, if we want to talk about a sequence of such things, we can get into trouble with where to put the indices: if we have $(\overline{a}_k)_{k<n}$, how do we talk about the individual coordinates of each element of the sequence of n? There are a couple of standard conventions: first of all, we can write the enumeration index as a superscript, rather than as a subscript (as long as we're not doing any arithmetic involving exponents!) and write $(\overline{a}_k)_{k<n}$, with the j-th coordinate n-tuple being $\left(a_j^k\right)_{k<n}$ or $\left\langle a_j^k \right\rangle_{k<n}$, or we can just remember which index means what, and write the j-th coordinate n-tuple as $(a_{k,j})_{k<n}$ or $\langle a_{k,j} \rangle_{k<n}($, with rule that the index in the n-tuple comes *first*, with the coordinate index j being applied to $\overline{a}_k$ to produce $a_{k,j}$.

So associated to any cartesian product are the *standard* or *coordinate* projection maps

$$\pi_j : \prod_{j\in J} A_j \to A_j : \overline{a} \mapsto a_j .$$

Note then that $\overline{a} = (\pi_j(\overline{a}))_{j\in J}$.

Cartesian product of algebras:

Let $\mathcal{L}$ be an algebraic language and $\mathcal{A} = \left\langle A; (f_i)_{i\in I}, (c_k)_{k\in k} \right\rangle$ be an abstract algebra for $\mathcal{L}$. If we want to talk about a family of algebras $(\mathcal{A}_j)_{j\in J}$ the subscripts and indexing that we would have to use would become overwhelming, so we adopt some shortcuts, by supressing the indexing of the operations and constants. So the underlying sets of the family of algebras will form a family of sets $(A_j)_{j\in J}$, and we will make statements like "let $\mathbf{f}$ be an operation of $\mathcal{L}$" (and introduce n as the arity of $\mathbf{f}$ without further comment), or "let $\mathbf{c}$ be a constant of $\mathcal{L}$". We use the necessary indexing into the family without specifically defining it.

Definition 0.1 *Let $(\mathcal{A}_j)_{j\in J}$ be a family of algebras for $\mathcal{L}$. We define an $\mathcal{L}$-structure $\mathcal{P} = \prod_{j\in J}\mathcal{A}_j$ on $\prod_{j\in J} A_j$ as follows:*

i. for each operation $\mathbf{f}$ of $\mathcal{L}$, $\mathbf{f}^{\mathcal{P}}(\overline{a}_1, \ldots, \overline{a}_n) = (f_j(a_{1,j}, \ldots, a_{n,j}))_{j\in J}$, with f_j of course being (more formally) $\mathbf{f}^{\mathcal{A}_j}$.

ii. for each constant $\mathbf{c}$ of $\mathcal{L}$, $\mathbf{c}^{\mathcal{P}} = (\mathbf{c}^{\mathcal{A}_j})_{j\in J}$.

Exercise Let $\mathbf{s} = \mathbf{t}$ be an $\mathcal{L}$-identity. Then the product $\mathcal{P}$ satisfies this identity iff each coordinate algebra $\mathcal{A}_j$ satisfies the identity.

If you can survive keeping track of notation, this is an easy exercise, and the proof is easier described informally: the operations on $\mathcal{P}$ are defined component-by-component, and so identities are determined component-by-component as well.

Standard homomorphisms and congruences.

Proposition 0.2 *Consider $\mathcal{P} = \prod_{j\in J}\mathcal{A}_j$.*

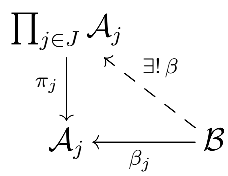
i. $\pi_j : \mathcal{P} \to \mathcal{A}_j$ is a surjective homomorphism.

ii. Θ_j defined by $\overline{a} \equiv \overline{b}(\Theta_j)$ iff $a_j = b_j$ is a congruence relation on $\mathcal{P}$, and $\mathcal{P}/\Theta_j \cong \mathcal{A}_j$.

Universal property

Theorem 0.3 *Consider $\mathcal{P} = \prod_{j\in J}\mathcal{A}_j$.*

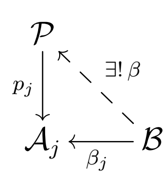
Let $\mathcal{B}$ be an $\mathcal{L}$-algebra and $(\beta_j : \mathcal{B} \to \mathcal{A}_j)$ be a family of homomorphisms. Then there is a unique homomorphism $\beta = \prod_{j\in J} p_j : \mathcal{B} \to \mathcal{P}$ such that for all $j \in J$, $\pi_j \circ \beta = \beta_j$.



Proof: $\beta(b) = (\beta_j(b))_{j\in J}$.

The fact that p is a homomorphism is immediate by the definition of product, and the uniqueness is just the obseervation that $\overline{a} = (\pi_j(\overline{a}))_{j\in J}$, made earlier. ∎

An immediate corollary of this (and any similar) universal property is that this diagram characterizes the product up to isomorphism, that is, if $(\mathcal{P}, (p_j)_{j\in J})$ also satisfies the diagram



then there is an isomorphism $\alpha : \mathcal{P} \cong \prod_{j\in J}\mathcal{A}_j$ with $\pi_j \circ \alpha = p_j$ for all $j \in J$.