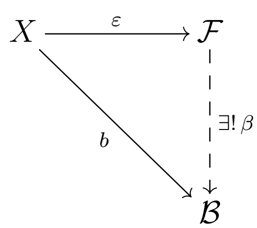
MATH 3322
Supplemental notes on Free Algebras

January 18, 2019

Definition 0.1 *Let $\mathcal{L}$ be an algebraic language and $\mathcal{K}$ a class of $\mathcal{L}$-algebras (closed under isomorphisms). [Later on, we will figure out the appropriate restrictions to put on $\mathcal{K}$.] Let X be a set, and $\varepsilon : X \to \mathcal{F}$ be an $\mathcal{L}$-algebra and a function mapping X into the underlying set F of $\mathcal{F}$.*

We say that $(\varepsilon, \mathcal{F})$ is a free algebra *on X in $\mathcal{K}$ if for all algebras $\mathcal{B} \in \mathcal{K}$ and maps $b : X \to B$, there is a unique homomorphism $\beta : \mathcal{F} \to \mathcal{B}$ such that $\varepsilon \circ \beta = b$.*



Theorem 0.2 *The free algebra on X in $\mathcal{K}$, if it exists, is unique up to isomorphism.*

Recall that the subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by $Y \subseteq A$ is the intersection of all subalgebras of $\mathcal{A}$ which contain Y (**Note:** we need an exercise on the equivalence of this with the "other" definition, where the underlying set B is found by "closing down" under the operations of the language.)

Proposition 0.3 *The free algebra $(\varepsilon, \mathcal{F})$ on X in $\mathcal{K}$, if it exists, is generated by $\varepsilon[X]$.*

Proof: Let $\mathcal{F}_0$ be the subalgebra of $\mathcal{F}$ generated by $\varepsilon[X]$. Then it is easy to see that $(\varepsilon, \mathcal{F}_0)$ is also free on X. So there is an isomorphism $\mathcal{F} \to \mathcal{F}_0$. But then the composition of this with the inclusion of $\mathcal{F}_0$ in $\mathcal{F}$ gives an automorphism of $\mathcal{F}$ which commutes with ε, which is therefore the identity. Therefore in fact the inclusion map is the identity ∎

Remark: This seems like an excessively wordy proof. Can anyone do better?

We will show one construction that proves that if $\mathcal{K}$ is closed under isomorphisms, products and subalgebras, then the free algebra in $\mathcal{K}$ on any set X exists.

We will also at least outline a different proof that the free algebra in any variety exists; and work out the details to show that the free group on a set X in any variety of groups exists. In particular, there are free groups on any set X, and free abelian groups on any set X.

Theorem 0.4 *Let $\mathcal{K}$ be closed under isomorphisms, products, and subalgebras. Then for any set X, the free algebra on X in $\mathcal{K}$ exists.*

Proof: The full details (with some diagram chasing!) will be given in class, but most of the proof is given here.

An X-*pair* [in $\mathcal{K}$] is any $(e, \mathcal{A})$, where $\mathcal{A} \in \mathcal{K}$, $e : X \to A$, and $\mathcal{A}$ is generated by $e[X]$. Two X-pairs $(e, \mathcal{A})$ and $(e', \mathcal{A}')$ are *isomorphic* if there is an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ such that $\psi \circ e = e'$. We write $\psi : (e, \mathcal{A}) \cong (e', \mathcal{A}')$.

Elementary counting arguments in set theory show first that there is an upper bound on the size of the underlying set of an X-pair, determined by the cardinality of X and the size of the language (in fact, $|\,A\,| \leq (|\,X\,| + \aleph_0)\,|\,\mathcal{L}\,|$). And again, simple counting arguments in set theory show that there are no more than $2^{|\,A\,|}$ automorphisms of a structure $\mathcal{A}$, so altogether there is no more than a set of equivalence classes of X-pairs.

That is, we can find an index set J and X-pairs $((e_j, \mathcal{A}_j))_{j \in J}$ such that for any X-pair $(e, \mathcal{A})$, there is $j \in J$ and an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}_j$ such that $\varphi \circ e = e_j$.

Let $\mathcal{P} = \prod_{j \in J} \mathcal{A}_j$ and define $\varepsilon : X \to P$ by $\varepsilon x = \langle\, e_j(x)\,\rangle_{j \in J}$. Let $\mathcal{F}$ be the subalgebra of $\mathcal{P}$ generated by $\varepsilon[X]$.

If $f : X \to A$ for some $\mathcal{A} \in \mathcal{K}$, then $f[X]$ generates a subalgebra $\mathcal{A}' \subseteq \mathcal{A}$, which is in $\mathcal{K}$ by assumption. Note that f is still a map from X into A', the underlying set of $\mathcal{A}'$! Then there is $j \in J$ and an isomorphism $\psi : (e_j, \mathcal{A}_j) \cong (f, \mathcal{A}')$. Define $\varphi : \mathcal{F} \to \mathcal{A}'$ by $\varphi = \psi \circ (\pi_j \restriction \mathcal{F})$. Then φ is naturally a map to $\mathcal{A}$ by the inclusion of $\mathcal{A}'$ in $\mathcal{A}$, and it is now easy to see that $\varphi \circ \varepsilon = f$. Furthermore, using the universal property of the direct product, we can see that φ is the unique such map. ∎

The term algebra

Treat X as a set of "letters"—new constant symbols—as an addition to the formal language $\mathcal{L}$. That is, the elements of X are distinct from all the symbols of $\mathcal{L}$, and are treated like constant symbols in forming terms. We denote the expanded language by $\mathcal{L}(X)$. A "closed term" is a term of $\mathcal{L}(X)$ that does not contain any variables. We denote the set of all closed terms by $\mathbb{T}_X$. [We really should include $\mathcal{L}$ in the notation as well, but there's no room!]

We define an $\mathcal{L}$-algebra $\mathcal{T} = \mathcal{T}_X$ on $\mathbb{T}_X$ by interpreting each constant symbol of $\mathcal{L}$ by itself—after all, a constant symbol is a closed term!—and by interpreting each operation symbol of $\mathcal{L}$ by the definition of compound terms: $\mathbf{f}^{\mathcal{T}}(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$, for closed terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$.

Theorem 0.5 *Let $\mathcal{K}$ be the class of* all *$\mathcal{L}$-algebras. Then $\mathcal{T}_X$ is the free algebra on X in $\mathcal{K}$.*

Proof: Exercise: This follows almost immediately by the definition of evaluation of terms in a structure. ∎

Theorem 0.6 *Let $\mathcal{V}$ be a non-trivial variety. Let $\Theta_{\mathcal{V}}$ be the meet of all the congruences Θ on $\mathcal{T}_X$ such that $\mathcal{F} = \mathcal{T}_X/\Theta \in \mathcal{V}$.*

Then $\mathcal{F}$ is the free algebra on X in $\mathcal{V}$.

Proof: To follow. ∎