MATH 3322

Supplemental notes on Finitely Generated Free Abelian Groups

A summary of the key formulas:

Theorem 0.1 *A subgroup G of a free abelian group F of rank n is a free abelian group of rank $m \leq n$. There is a basis [free generating set] $\{u_1, \ldots, u_n\}$ of F and integers $t_1, \ldots, t_m \geq 1$ such that (1) $\{t_1u_1, \ldots, t_mu_m\}$ is a basis for G and (2) $t_i | t_{i+1}$ for each i, $1 \leq i < m$.*

(1) $v_1 = t_1x_1 + k_2x_2 + \cdots + k_nx_n$ is chosen with $v_1 \in G$, $\{x_1, \ldots, x_n\}$ a basis for F, and t_1 minimal amongst all such possible combinations.

We are doing an induction on the rank n, and have noted that the result is immediate for $n = 1$. So we assume in the following that the result holds for all free abelian groups of rank $< n$.

(1a) We showed that $t_1 | k_j$ for each j, that is, $k_j = q_jt_1$ for each j. It follows that if we set $u_1 = x_1 + \sum_{j-2}^{n} q_jx_j$ then $\{u_1, x_2, \ldots, x_n\}$ is a basis for F, and $v_1 = t_1u_1$.

(2) Again since $\{x_1, \ldots, x_n\}$ is a basis for F, for any $y \in G$ we can find integers so that $y = h_1x_1 + \cdots + h_nx_n$. We proved that $t_1 | h_1$, that is, $h_1 = qt_1$ for some q.

(3) Then from (1),

$$y - qv_1 = (h_1 - qt_1)x_1 + (h_2 - qk_2)x_2 + \cdots + (h_n - qk_n)x_n .$$

But $h_1 - qt_1 = 0$.

So by bringing qv_1 to the right hand side of the equation and renaming all the coefficients, we find that any $y \in G$ can be expressed in the form

$$y = s_1v_1 + s_2x_2 + \cdots s_nx_n = s_1(t_1u_1) + s_2x_2 + \cdots s_nx_n .$$

(4) Observe that since $\{x_1, \ldots, x_n\}$ is a free basis for F, $\{x_2, \ldots, x_n\}$ is a basis for a free abelian group H of rank $n - 1$.