MATH 3322 Problem Set 5

March 8, 2019

Solutions

[3] **Question 1.** Prove that if H is normal in G and both H and G/H are solvable, then so is G.

Proof: Let

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_m = H$$

and

$$\{1\} = G_0/H \leq G_1/H \leq \cdots \leq G_N/H = G/H$$

be subnormal series with abelian factors. Recall that every subgroup of G/H has the form G'/H for some G', $H \leq G' \leq G$, and that $G'/H \trianglelefteq G/H$ iff $G' \trianglelefteq G$. So

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_m = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

is a subnormal series in G, and furthermore since $(G_{i+1}/H)/(G_i/H) \cong G_{i+1}/G_i$ and the former is abelian, this series has abelian factors. Hence G is solvable. ∎

[2] **Question 2.** Prove that if H is normal in G and $H \leq K \leq G$, then $K/H \subseteq Z(G/H)$ iff $[K, G] \subseteq H$.

Proof: For $k \in G$, $k/H \in Z(G/H)$ iff for all $g \in G$, $(k^{-1}/H)(g^{-1}/H)(k/H)(g/H) = H$ iff for all $g \in G$, $k^{-1}g^{-1}kg = [k, g] \in H$. Therefore for $H \leq K \leq G$, $K/H \subseteq Z(G/H)$ iff $[K, G] \subseteq H$. ∎

Remark: For the original version of the question, note that if $H \leq K_1 < K \leq G$ and $[K, G] \subseteq H$, it is still true that $[K_1, G] \subseteq H$, but clearly $K_1/H \subset K/H$.

[2] **Question 3.** Prove that if A is a free abelian group on S and B is a free abelian group on T and $S \cap T = \emptyset$, then $A \oplus B$ is a free abelian group on $S \cup T$.

Proof: Suppose that H is an abelian group and $h : S \cup T \to H$. Since $S \cap T = \emptyset$, $h = (h \restriction S) \cup (h \restriction T$. Then there are unique homomorphisms $h_A : A \to H$ and $h_B : B \to H$ such that $h_A \restriction S = h \restriction S$ and $h_B \restriction T = h \restriction T$ by the universal property of abelian groups. Then by the universal property of the direct sum, it follows that $h_A \oplus h_B : A \oplus B \to H$ defined by $(h_A \oplus h_B)(\langle a, b \rangle) = h_A(a) + h_B(b)$ is the unique homomorphism $A \oplus B \to H$ extending h. ∎

Remark: There is a nice diagram at the end. Question 4 follows *after* Question 5

. . . and one of you found the "sneaky" proof:

Proof:

$A \cong \mathbb{Z}^{(S)}$, $B \cong \mathbb{Z}^{(T)}$, and since $S \cap T = \emptyset$, $A \oplus B \cong \mathbb{Z}^{(S \cup T)}$, the free group on $S \cup T$. ∎

If I use this question again in the future, I will have to say "prove *from the definitions*. . . "

Question 5. An abelian group A is called *divisible* if for every $a \in A$ and every $0 \neq n \in \mathbb{Z}$, there is $b \in A$ such that $nb = a$.

Clearly $\langle \mathbb{Q}; +, -, 0 \rangle$ is divisible.

[2] (a) Prove that a finite abelian group is not divisible.

Hint: Consider an element of prime power order.

Remark: A group G is *of bounded exponent* if for some n, $g^n = 1$ for all $n \in G$. Your proof will (likely) show in fact that an abelian group of bounded exponent is not divisible.

Further Remark: As several of you have pointed out, the Remark is a better hint than the Hint.

Proof: Suppose G is a finite abelian group, $|G| = n$. Then for every $b \in G$, $nb = 0$. So if $0 \neq a \in G$, a is not divisible by n.

In fact, if G is an abelian group of exponent n, then every $b \in G$, $nb = 0$, so the same proof applies. ∎

[3] (b) Prove that an (arbitrary) direct sum of abelian groups is divisible iff each of the summands is divisible.

Proof: Let $A = \bigoplus_I A_i$ be a direct sum of abelian groups.

Really the proof boils down to nothing more than noting that for $\overline{b} = \langle b_i \rangle_{i \in I} \in A$, $n\overline{b} = \langle nb_i \rangle_{i \in I}$.

If A is divisible, $c \in A_i$, and $0 \neq n \in \mathbb{Z}$, let $\overline{a} \in A$ be any tuple with $a_i = c$, find $\overline{b} \in A$ such that $n\overline{b} = \overline{a}$. Then $nb_i = c$.

On the other hand, if each A_i is divisible, and $\overline{a} \in A$, for each $i \in I$ such that $a_i \neq 0$, find $b_i \in A_i$ such that $nb_i = a_i$ and otherwise set $b_i = 0$. Then $n\overline{b} = \overline{a}$. ∎

[2] (c) Prove that a homomorphic image of a divisible abelian group is divisible.

Proof: Suppose $\psi : A \to C$ is a surjective homomorphism of abelian groups. Let $c \in C$ and $0 \neq n \in \mathbb{Z}$. Then there is $a \in A$ such that $\psi(a) = c$ and $b \in A$ such that $nb = a$. Then $n\psi(b) = \psi(a) = c$. ∎

[3] (d) Prove that $\mathbb{Q}/\mathbb{Z}$ is a divisible group in which every element is of finite order, and in which there are elements of any finite order.

Proof: $\mathbb{Q}$ is a divisible abelian group so $\mathbb{Q}/\mathbb{Z}$ is divisible by part (c). If $a/b \in \mathbb{Q}$ is a fraction in lowest form with $b > 1$, (so $(a/b) + \mathbb{Z} \neq 0$) then $b(a/b) \in \mathbb{Z}$ but $c(a/b) \notin \mathbb{Z}$ for any c, $1 \leq c < a$), and so the order of $a/b + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ is b. ∎

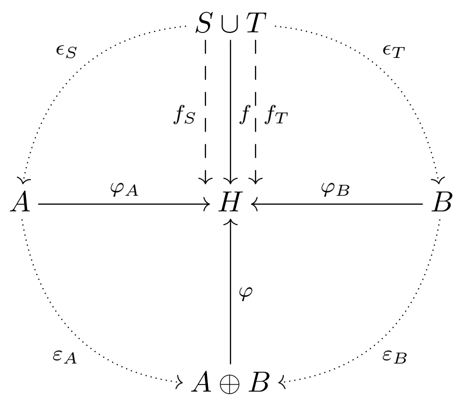
[3] **Question 4.** Prove (from the definition) that a homomorphic image of a nilpotent group is nilpotent.

Proof: Let $0 = Z_0 < Z_1 < \cdots < Z_n = G$ be the ascending central series of the group G, $\psi : G \twoheadrightarrow H$ an epimorphism,, and $0 = Z'_0 < Z'_1 < \cdots < Z'_m \leq H$ the ascending central series in H. So in particular for $i < n$ (or for $i < m$, as the case may be), Z_{i+1} $[Z'_{i+1}]$ is the full inverse image in G [in H] of $Z(G/Z_i)$ [of $Z(H/Z'_i)$.

I claim that for $i \leq n$, $\psi[Z_i] \subseteq Z'_i$, and so in particular $Z'_i = H$ for some $i \leq n$.

I procede by induction on $i \leq n$. Clearly if $h \in Z_1 = Z(G)$ then $\psi(h)$ commutes with every element of H. If we have shown that $\psi[Z_i] \subseteq Z'_i$ and $h \in G$ is such that hZ_i commutes with every element of G/Z_i, then for all $g \in G$, $g^{-1}h^{-1}gh \in Z_i$, so $\psi(g^{-1}h^{-1}gh) \in \psi[Z_i] \subseteq Z'_i$. But ψ is a surjection, so $\psi(h)$ is such that $\psi(h)Z'_i$ commutes with every element of H/Z'_I, that is, $\psi(h) \in Z'_{i+1}$. ∎

Question 3.



$\langle S, \epsilon_S, A \rangle$ and $\langle S, \epsilon_S, A \rangle$ are free abelian group diagrams.

$\langle A, \varepsilon_A, B, \varepsilon_B, A \oplus B \rangle$ is a direct sum diagram.

We define $\epsilon : S \cup T \to A \oplus B$ by $\epsilon = \varepsilon_A \circ \epsilon_S \cup \varepsilon_B \circ \epsilon_T$. ϵ is well-defined since $S \cap T = \emptyset$.

$f : S \cup T \to H$ is a test group for the definition of "$\langle S \cup T, \epsilon, A \oplus B \rangle$ is a free abelian group".

The maps f_S and f_T are defined by restriction.

φ_A and φ_B exist and are unique making the upper left quadrant and the upper right quadrant commutative, respectively, since A and B are free.

φ exists and is unique by the universal property of the direct sum.

Then by the definitions, and for $s \in S$, $\varphi\epsilon(s) = \varphi\varepsilon_A\epsilon_S(s) = \varphi_A\epsilon_S(s) = f_S(s)$, and similarly for $t \in T$. And so (again since S and T are disjoint), $\varphi\epsilon = f$.