MATH 3322 Problem Set 3

February 11, 2019

Solutions

I couldn't think of anything better than to ask you to "finish off" some of the things that I only hinted at in the lecture about finitely presented abelian groups.

Homomorphisms between finitely generated abelian groups.

[6] **Question 1.** Consider be elements of $\mathbb{Z}^m$ (etc) as being represented by *row* vectors of integers. Let $\mathbf{e}_i$ be the i-th standard basis vector.

(a) Let A be an $m \times n$ matrix over the integers. Show that the map

$$\varphi_A : \mathbb{Z}^m \to \mathbb{Z}^n : \overline{a} \mapsto \overline{a}A$$

defines an abelian group homomorphism.

Proof: If $\overline{a}$, $\overline{b} \in \mathbb{Z}^m$ then $(\overline{b} - \overline{a})A = \overline{b}A - \overline{a}A$ by the usual rules for matrix multiplication. So φ_A is a homomorphism of abelian groups.

Remark In fact, in a more general context it is a homomorphism of left $\mathbb{Z}$-modules. All that this means is that it respects scalar multiplication by integers as well: If $z \in \mathbb{Z}$, then $(z\overline{a})A = z(\overline{a}A)$, which of course is another elementary property of matrix multiplication. ∎

(b) Suppose that $\psi : \mathbb{Z}^m \to \mathbb{Z}^n$ is an abelian group homomorphism. Define an $m \times n$ matrix over the integers $M(\psi)$ by setting the i-th row of $M(\psi)$ to be $\psi(\mathbf{e}_i)$. Show that for all $\overline{a} \in \mathbb{Z}^m$, $\psi(\overline{a}) = \overline{a}M(\psi)$.

Proof: Let $\overline{a} = \sum_{i=1}^m z_i \mathbf{e}_i \in \mathbb{Z}^m$.

Then $\psi(\overline{a}) = \sum_{i=1}^m z_i \psi(\mathbf{e}_i)$ and $\overline{a}M(\psi) = \sum_{i=1}^m z_i \mathbf{e}_i M(\psi)$. So it suffices to show that $\psi(\mathbf{e}_i) = \mathbf{e}_i M(\psi)$. But $\mathbf{e}_i M(\psi)$ is the i-th row of $M(\psi)$, so the equality holds by definition ∎

(c) Verify that $\varphi_{M(\psi)} = \psi$ and $M(\varphi_A) = A$.

Proof: It suffices to check the first equation on the standard basis vectors.

$$\varphi_{M(\psi)}(\mathbf{e}_i) = \mathbf{e}_i M(\psi) = \psi(\mathbf{e}_i)\,.$$

It suffices to check the second equation row-by-row. The i-th row of $M(\varphi_A)$ is $\varphi_A(\mathbf{e}_i) = \mathbf{e}_i A$, that is, the i-th row of A. ∎

Finitely presented abelian groups.

Definition 0.1 *A* finite presentation *of an abelian group M consists of a tuple $\langle\, \varepsilon,\, X,\, \Sigma\,\rangle$ where X is a finite set of variables, $\varepsilon : X \to M$, and Σ is a finite set of abelian group words in X, such that each word $w \in \Sigma$ evaluates to 0 in M, and M is "free" with respect to this property. That is, if B is any other abelian group and $b : X \to B$ such that every word $w \in \Sigma$ evaluates to 0 in B, then there is a unique group homomorphism $\beta : M \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\varepsilon"] \arrow[dr, "b"'] & M \arrow[d, dashed, "\exists!\,\beta"] \\ & B \end{tikzcd}$$

It is normal to suppress the map ε in the notation and treat X as if it were a subset of M.

For instance, it is not hard to see that $\langle \{ a, b \}, \{ 4a, 6b \} \rangle$ is a presentation of the group $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. It is more traditional to write this presentation as something like $\langle a, b : 4a = 6b = 0 \rangle$.

Let $\langle \varepsilon, \mathcal{F} \rangle$ be the free abelian group on X and Θ the intersection of all congruences Ψ on $\mathcal{F}$ such that $w \equiv 0(\Psi)$ for each $w \in \Sigma$, and $\natural$ the quotient map $\mathcal{F} \to \mathcal{F}/\Theta$.

[4] **Question 2.** Prove that $\langle \natural \circ \varepsilon, X, \mathcal{F}/\Theta \rangle$ is the group finitely presented by Σ.

Proof: So we have to show that the given data satisfies the appropriate universal diagram.

$$\begin{tikzcd} X \arrow[r, "\varepsilon"] \arrow[drr, "f"'] & \mathcal{F} \arrow[r, "\natural"] \arrow[dr, dotted, "\overline{f}"] & \mathcal{F}/\Theta \arrow[d, dashed, "\varphi"] \\ & & A \end{tikzcd}$$

We are given $f : X \to A$ where A is an abelian group such that every word in Σ evaluates to 0 in A, and we have to find a homomorphism φ completing the diagram as shown.

But we get a homomorphism $\overline{f}$ since $\mathcal{F}$ is free on X. Since every word in Σ evaluates to 0 in A, $\ker(\overline{f}) \supseteq \Theta$, and so $\overline{f}$ factors through $\mathcal{F}/\Theta$, as required. ∎

Abstract presentations

We don't have the proper context in which to give this definition. Just assume that $\mathcal{V}$ is a variety in which there is an algebra 0 and it has "all the right properties". We will only be working in the variety of abelian groups, where this is indeed the case.

Definition 0.2 *An algebra $\mathcal{A}$ in a variety $\mathcal{V}$ is* finitely presented *if there is an* exact sequence

$$\mathcal{F}_m \xrightarrow{\psi} \mathcal{F}_n \xrightarrow{\varphi} \mathcal{A} \longrightarrow 0$$

where $\mathcal{F}_m$ and $\mathcal{F}_n$ are the free algebras in $\mathcal{V}$ on m and n generators respectively, and exact *means that at each algebra in the sequence the image of the incoming homomorphism is the kernel of the outgoing homomorphism. That is,* $\mathrm{im}(\psi) = \ker(\varphi)$ *and* $\mathrm{im}(\varphi) = \mathcal{A}$ *: φ is a surjection.*

The purpose of this exercise is to help you see that the two definitions of finitely presented abelian groups that we have given are equivalent.

But to start off:

[2] **Question 3.** Show that every finitely generated abelian group is finitely presented.

Proof: Suppose A is an n-generated abelian group. Then there is a surjection φ from the free abelian group $\mathcal{F}_n$ onto A. Let $K \leq \mathcal{F}_n$ be the kernel of φ. Then since K is a subgroup of a finitely generated free abelian group, K is also free abelian, say on m generators. So there is an isomorphism ψ of $\mathcal{F}_m$ onto K, thus

$$\mathcal{F}_m \xrightarrow{\psi} \mathcal{F}_n \xrightarrow{\varphi} \mathcal{A} \longrightarrow 0$$

is a finite presentation of A. ∎

Hint: In any variety whatsoever, every finitely generated algebra is the image of a finitely generated free algebra. Use the theorem that a subgroup of a finitely generated free abelian group is finitely generated free abelian.

Note that a finite set Σ of group words (say in variables $x_1, \dots x_n$) is just a finite set of integer linear combinations of those variables; and if we assert that those linear combinations are all 0, we are really just asserting a matrix equation $A\overline{x} = 0$ for an $m \times n$ integer matrix, where m is the number of equations. Let ψ_A be the homomorphism defined in 1(a), $K = \operatorname{im}(\psi_A) \subseteq \mathbb{Z}^n$, and $M = \mathbb{Z}^n / K$.

Consider the sequence of homomorphisms

$$\mathbb{Z}^{(m)} \xrightarrow{\psi_A} \mathbb{Z}^{(n)} \xrightarrow{\natural} M \longrightarrow 0$$

Let $m_1, \dots, m_n$ be the images in M of the standard basis vectors of $\mathbb{Z}^n$, and $\overline{m}$ the column vector of these elements.

[8] **Question 4.** Show that in M, $A\overline{m} = 0$.

Proof: If you see the "fast" way of writing out the explanation, the explanation itself is barely worth 2 or 3 marks. The 8 points reflect the difficulty of seeing your way through to the end, *not* the difficulty of the "most efficient solution".

$$A\overline{m} = A \begin{bmatrix} \natural(\mathbf{e}_1) \\ \vdots \\ \natural(\mathbf{e}_n) \end{bmatrix} = A\natural(I_n) = \natural(AI_n) = \natural(I_m A) = 0\,,$$

The first few equalities are all just matrix manipulations, and the final step follows since the rows of $I_m A$ are by definition in the kernel of $\natural$. ∎

Hints over...

Hints: On the one hand this is trivial. On the other hand, it requires some real "thinking outside the box" to get the notation right.

Here is a rather rambling discussion of relevant matters. In the end, with the proper understanding of the meaning of the components, I can write down the equations that prove this result in one line: but you need to write out some things to explain where the equations come from. All of that occurs or is hinted at in what follows:

Let I_m, I_n be the respective identity matrices.

First consideration: Row i of A is $\psi_A(\mathbf{e}_i)$. Imagine stacking up the standard basis vectors. They make I_m. So if we think of stacking up the images of ψ_A as rows of $\mathbb{Z}^n$, all we are doing is looking at $I_m A$, that is, A. And since the standard basis vectors of $\mathbb{Z}^m$ generate $\mathbb{Z}^m$, the rows of A generate the image K of ψ_A.

Now the natural map sends each standard basis vector $\mathbf{e}_i$ of $\mathbb{Z}^n$ to the corresponding generator m_i of M. Stacking those up again gives us $\natural(I_n) = \overline{m}$.

It should be fairly clear (and you can use it with out proof) that if θ is any abelian group homomorphism, A an integer matrix, and $\bar{b}$ a column vector of elements of the correct length, then $\theta(A\bar{b}) = A\theta(\bar{b})$, coordinate by coordinate, and similarily, with multiplication on the other side, for row vectors.

Stringing all these ideas together in the proper order, you can then prove the required result, that $A\overline{m} = 0$. I'm looking for a "clean" exposition with the right calculations.

(It follows from Questions 3 and 4 with only a little extra work that then, once again, M is the abelian group presented by $\langle X, \Sigma \rangle$.)

[20] TOTAL