MATH 3322 Problem Set 2

February 5, 2019

Solutions

[6] **Question 1.** A group $\langle G; *, ^{-1}, \mathbf{e} \rangle$ is *of exponent* n if for all $g \in G$, $x^n = \mathbf{e}$. Equivalently, for an abelian group A, $na = 0$ for all $a \in A$.

$\mathbb{Z}_n$ is shorthand for the group of integers modulo n, more correctly written as $\mathbb{Z}/n\mathbb{Z}$. We take the elements of $\mathbb{Z}_n$ to be $\{0, 1, \ldots, n-1\}$.

[1] (a) Let $\langle A; +, -, 0 \rangle$ be an abelian group and $a \in A$ such that $na = 0$. Prove that there is a unique homomorphism $\varphi : \mathbb{Z}_n \to A$ such that $\varphi(1) = a$.

Proof: Note that in $\mathbb{Z}_n$, $k = k \times 1$. So if we want a homomorphism such that $\varphi(1) = a$, then necessrily $\varphi(k) = ka$. Now for any integers z, z', if $z \times 1 = z' \times 1$ in $\mathbb{Z}_n$, then $n|(z - z')$ and so in A, $(z - z')a = 0$. Thus φ is well-defined. It is clearly a homomorhism.

Note: if $na \neq 0$, the map described is not well-defined. ∎

[5] (b) Let X be any non-empty set. Consider the *weak direct power* $\mathbb{Z}_n^{(X)}$. [For details on weak direct powers and direct sums, see the next exercise.] Define $\varepsilon : X \to \mathbb{Z}_n^{(X)}$ by $\varepsilon(x)_x = 1$, $\varepsilon(x)_y = 0$ for $y \neq x$, $y \in X$.

Prove that $(\varepsilon, \mathbb{Z}_n^{(X)})$ is the free abelian group of exponent n on X.

Proof: Suppose that $h : X \to A$, where A is an abelian group of exponent n. Then in A, $nh(x) = 0$ for all $x \in X$.

Therefore by part (a), there is a unique, well-defined, group homomorphism from the x-th component of $\mathbb{Z}_n^{(X)}$ to A, sending the generator (that is, $\varepsilon(x)$) to $h(x)$ given by $k\varepsilon(x) \mapsto kh(x)$.

Every element of $\mathbb{Z}_n^{(X)}$ is (uniquely) a finite sum of multiples of the generators, $a = \sum_{x \in X} k_x \mathbf{e}_x$, where $k_x = 0$ almost everywhere, and without loss of generality, $0 \leq k_x < n$. Thus there is only one way to define a homomorphism extending h: $\sum_{x \in X} k_x \mathbf{e}_x \mapsto \sum_{x \in X} k_x h(x)$. ∎

[8] **Question 2.** Let $(A_i)_{i\in I}$ be a non-empty family of abelian groups. Let

$$\bigoplus_{i\in I} A_i = \left\{ \overline{a} \in \prod_{i\in I} A_i : a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

For each $i \in I$, let $\varepsilon_i : A_i \to \bigoplus_{i\in I} A_i$ be defined by

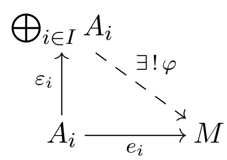
$$\varepsilon_i(a)_j = \begin{cases} a & j = i \\ 0 & j \neq i \end{cases}, \text{ for each } a \in A_i\,.$$

$(\bigoplus_{i\in I} A_i,\ (\varepsilon_i)_{i\in I})$ is called the *direct sum* of the abelian groups $(A_i)_{i\in I}$. If all the A_i are equal to a single group A, we write it as $A^{(I)}$, and call it the *weak direct power* of A.

[2] (a) Prove that $\bigoplus_{i\in I} A_i$ is a subgroup of $\prod_{i\in I} A_i$, and that each ε_i is a group homomorphism.

Proof: Operations on the direct sum or direct product are defined component-by-component. Clearly if $\overline{a}$ and $\overline{b}$ are zero on all but a finite number of components, so is $\overline{a} - \overline{b}$. ε_i sends A_i to elements which are non-zero on at most component i, so operations on the images are just the operations in A_i, so ε_i is a homomorphism. ∎

[6] (b) Prove that $\bigoplus_{i\in I} A_i$ satisfies the universal property described by the following diagram:



where M is any abelian group and $e_i : A_i \to M$ is a family of group homomorphisms.

[For clarity: This means that we are given the diagram with all components i simultaneously, and complete it by a single map φ.]

Proof: Note that 1(b) and 2(b) involve essentially the same argument.

Every element $\overline{a}$ of the direct sum has $\overline{a}_i = 0$ for all but finitely many $i \in I$. Therefore every element of the direct sum can be written in one and only one way as a (finite) sum of non-zero elements of the images of the various maps ε_i. Any homomorphism ψ from the direct sum to M has to respect these sums, $\psi(\overline{a}) = \sum_{t=1}^{n} \psi(a_{i_t})$, where $i_1, \ldots, i_n$ are the coordinates on which $\overline{a}$ is non-zero. [But notice what we have done here: we have identified/confused an element $a \in A_i$ with its image $\varepsilon_i(a)$ in the direct sum. This is "standard practice", and reduces the overhead of notation.]

So, given the maps e_i, there is one and only one way to define φ completing the diagram, $\varphi(\overline{a}) = \sum_{t=1}^{n} e_{i_t}(a_{i_t})$. The result is clearly a homomorphism as that only depends on addition working coordinate by coordinate.

If you want a more careful treatment of "where things are", remember that the projection maps are homomorphisms, and that $\overline{a} = \langle\, \pi_i(\overline{a}) \,\rangle_{i\in I}$. So a more formal definition of φ would be

$$\varphi(\overline{a}) = \sum_{i\in I} e_i(\pi_i(\overline{a}))\,,$$

the sum being well-defined since $\pi_i(\overline{a})$ is zero almost everywhere. ∎

[6] **Question 3.** Let $(G_i)_{i\in I}$ be a non-empty family of groups. Let

$$\prod_{i\in I}^{W} G_i = \left\{ \overline{a} \in \prod_{i\in I} G_i : a_i = \mathbf{e}_i \text{ for all but finitely many } i \in I \right\}.$$

Let $\varepsilon_i : G_i \to \prod_{i\in I}{}^{W} G_i$ be defined by

$$\varepsilon_i(a)_j = \begin{cases} a & j = i \\ \mathbf{e}_j & j \neq i \end{cases}$$

Just as in Question 2, it is easy to see that $\prod_{i\in I}{}^{W} G_i$ is a subgroup of $\prod_{i\in I} G_i$, and that each ε_i is an embedding. You can assume this result without proof.

[2] (a) Show that $\prod_{i\in I}^{W} G_i$ is a *normal* subgroup of $\prod_{i\in I} G_i$.

Proof: Let $\overline{a} \in \prod_{i\in I}^{W} G_i$ and $\overline{b} \in \prod_{i\in I} G_i$. We need to show that $\overline{b}^{-1}\overline{a}\overline{b} = \mathbf{e}$.

But $\overline{a}$ is the identity almost everywhere. So on any coordinate i on which $\overline{a}$ is the identity, $(\overline{b}^{-1}\overline{a}\overline{b})_i = b_i^{-1}\mathbf{e}b_i = \mathbf{e}$. Therefore $\overline{b}^{-1}\overline{a}\overline{b}$ is the identity almost everywhere. ∎

[1] (b) Show that for any group G and any $g \in G$, the map $\psi : \mathbb{Z} \to G : z \mapsto g^z$ is a group homomorphism that sends $1 \in \mathbb{Z}$ to g.

Proof: $\psi(z + w) = g^{z+w} = g^z g^w$, so ψ is a homomorphism. ∎

[This shows that $\mathbb{Z}$ is not just the free abelian group on one generator, but in fact it is the free group on one generator.]

[3] (c) Show that if G is a group with $a, b \in G$ such that $ab \neq ba$, then the pair of homomorphisms $e_1 : \mathbb{Z} \to z \mapsto a^z$, $e_2 : \mathbb{Z} \to z \mapsto b^z$, does not lift to a homomorphism $\mathbb{Z} \times \mathbb{Z} \to G$.

Proof: Implicit here is that e_1 is acting on the first component of $\mathbb{Z} \times \mathbb{Z}$, and e_2 on the second. $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ commute in $\mathbb{Z} \times \mathbb{Z}$, but a and b do not commute in G. Homomorphisms preserve identities, so a and b cannot be the images of $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ ∎

[This shows that the weak direct product of groups does not satisfy (for the class of all groups) the universal property described in 2(b), even for a product of only two groups! In particular, $\mathbb{Z}^2$ is *not* the free group on two generators.]

Remark: In abstract classes of algebras, the universal property 2(b) defines an object called the *free product*. Abelian groups (and more generally modules over a ring) are unusual in that the free product has a simple description by the direct sum. One of the basic properties of the free product is that if $\mathcal{F}_1$ is the free algebra on one generator in a variety $\mathcal{V}$, then the free product of X copies of $\mathcal{F}_1$ is the free algebra on X.

[20] TOTAL