MATH 3322 Problem Set 1

January 29, 2019

Solutions

Recall the following definitions and notation from the handout on Universal Algebra.

An *algebraic language* $\mathcal{L}$ is determined by a set $(\mathbf{f}_i)_{i\in I}$ of *function symbols*, each $\mathbf{f}_i$ a a $\nu(i)$-ary function symbol (where $\nu : I \to \omega \setminus \{0\}$); and a set $(\mathbf{c}_k)_{k\in k}$ of *constant* symbols.

An *abstract algebra* $\mathcal{A}$ *for* $\mathcal{L}$ consists of a non-empty set A and actual operations and elements on A interpreting the symbols of $\mathcal{L}$.

An *assignment of values* in $\mathcal{A}$ is a map α : variables $\to A$.

You will need to review and refer to the definitions of *subalgebra*, *homomorphism*, and *congruence*, and Definition 0.4, how to evaluate a term in an algebra.

Here is the example from class:

Lemma 0.1 *Let $\mathcal{B} \subseteq \mathcal{A}$ be $\mathcal{L}$-algebras, and $\mathbf{t}$ an $\mathcal{L}$-term, α an assignment in $\mathcal{A}$. Then*

$$\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{t}^{\mathcal{A}}[\alpha]$$

Proof: If $\mathbf{t}$ is a variable v, then $\mathbf{t}^{\mathcal{B}}[\alpha] = \alpha(v) = \mathbf{t}^{\mathcal{A}}[\alpha]$.

If $\mathbf{t}$ is a constant symbol $\mathbf{c}$, then $\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{c}^{\mathcal{B}} = \mathbf{c}^{\mathcal{A}} = \mathbf{t}^{\mathcal{A}}[\alpha]$; the first and last equalities by the definition of evaluation, and the middle by the definition of subalgebra.

If $\mathbf{t}$ is a compound term $\mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ and the Lemma holds for $\mathbf{t}_1, \ldots, \mathbf{t}_n$, then

$$\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{f}^{\mathcal{B}}({\mathbf{t}_1}^{\mathcal{B}}[\alpha], \ldots, {\mathbf{t}_n}^{\mathcal{B}}[\alpha]) = \mathbf{f}^{\mathcal{A}}({\mathbf{t}_1}^{\mathcal{A}}[\alpha], \ldots, {\mathbf{t}_n}^{\mathcal{A}}[\alpha]) = \mathbf{t}^{\mathcal{A}}[\alpha];$$

the first and last equalities by the definition of evaluations, and the middle one by the definition of subalgebra (for $\mathbf{f}$) and the assumption on the terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$. ∎

[4] **Question 1.** Let $\mathcal{B} \subseteq \mathcal{A}$ be $\mathcal{L}$-algebras.

(a) Prove that if an identity $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$, then the identity also holds in $\mathcal{B}$.

Proof: Observe that since $B \subseteq A$, an assignment in $\mathcal{B}$ is automatically an assignment in $\mathcal{A}$. So for any assignment α in $\mathcal{B}$,

$$\mathbf{s}^{\mathcal{B}}[\alpha] = \mathbf{s}^{\mathcal{A}}[\alpha] = \mathbf{t}^{\mathcal{A}}[\alpha] = \mathbf{s}^{\mathcal{B}}[\alpha],$$

the first and third inequalities by the lemma in the introduction, and the middle equality because the identity holds in $\mathcal{A}$. Hence the identity holds in $\mathcal{B}$. ∎

(b) Give a simple example in groups to show that the converse does not hold.

Solution: Note that *every* identity holds in the trivial group which is a subgroup of every group. But clearly there are groups satisfying non-trivial identities!

Question 2. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of $\mathcal{L}$-algebras, $\mathbf{t}$ an $\mathcal{L}$-term, and α an assignment in $\mathcal{A}$.

[3] (a) Prove that

$$\varphi(\mathbf{t}^{\mathcal{A}}[\alpha]) = \mathbf{t}^{\mathcal{B}}(\varphi \circ \alpha) .$$

Proof: By induction on the complexity of $\mathbf{t}$. If $\mathbf{t}$ is a variable the result is immediate by the definition of the assignment $\varphi \circ \alpha$, if $\mathbf{t}$ is a constant symbol, the result is immediate by the definition of homomorphism, and if $\mathbf{t}$ is a compound term $\mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ where (a) hold for $\mathbf{t}_1, \ldots, \mathbf{t}_n$, then

$$\begin{aligned} \varphi(\mathbf{t}^{\mathcal{A}}[\alpha]) &= \varphi(\mathbf{f}^{\mathcal{A}}(\mathbf{t}_1^{\mathcal{A}}[\alpha], \ldots, \mathbf{t}_n^{\mathcal{A}}[\alpha])) \\ &= \mathbf{f}^{\mathcal{B}}(\varphi(\mathbf{t}_1^{\mathcal{A}}[\alpha]), \ldots, \varphi(\mathbf{t}_n^{\mathcal{A}}[\alpha])) \\ &= \mathbf{f}^{\mathcal{B}}(\mathbf{t}_1^{\mathcal{B}}[\varphi \circ \alpha], \ldots, \mathbf{t}_n^{\mathcal{B}}[\varphi \circ \alpha]) \\ &= \mathbf{t}^{\mathcal{A}}[\varphi \circ \alpha] \end{aligned}$$

∎

[3] (b) Prove that if φ is surjective, and if an identity $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$, then the identity also holds in $\mathcal{B}$.

Proof: Suppose that φ is surjective and the identity $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$.

Let β be an assignment in $\mathcal{B}$. [We need to verify that $\mathbf{s}^{\mathcal{B}}[\beta] = \mathbf{t}^{\mathcal{B}}[\beta]$.] Since φ is surjective, for each variable v and $\beta(v) = b \in B$, we can find $a \in A$ such that $\varphi(a) = b$. Thus we can define an assignment α in A so that for all variables v, $\varphi(\alpha(v)) = \beta(b)$, that is, $\varphi \circ \alpha = \beta$.

Therefore

$$\mathbf{s}^{\mathcal{B}}[\beta] = \mathbf{s}^{\mathcal{B}}[\varphi \circ \alpha] = \varphi(\mathbf{s}^{\mathcal{A}}[\alpha]) = \varphi(\mathbf{t}^{\mathcal{A}}[\alpha]) = \mathbf{t}^{\mathcal{B}}[\varphi \circ \alpha] = \mathbf{t}^{\mathcal{B}}[\beta] .$$

The second and fourth equalities hold since φ is a homorphism, and the middle equality since $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$. ∎

[2] (c) Find a simple example in groups to show that the converse to (b) does not hold.

Solution: Same answer as 1(b): every identity holds in the trivial group; and furthermore, the identity is a homomorphic image of every group.

[8] **Question 3.** Let $(\mathcal{A}_i)_{i\in I}$ be a family of $\mathcal{L}$-algebras and $\mathcal{P} = \prod_{i\in I} \mathcal{A}_i$.

Define relations $(\Theta_i)_{i\in I}$ on $\mathcal{P}$ by $\overline{a} \equiv \overline{b}(\Theta_i)$ iff $a_i = b_i$. It's obvious that each Θ_i is an equivalence relation, and you don't have to prove this.

(a) Prove that each Θ_i is a congruence relation.

Proof: If $\mathbf{f}$ is n-ary and $\overline{a}_1, \ldots, \overline{a}_n, \overline{a}'_1, \ldots, \overline{a}'_n \in \mathcal{P}$ are such that $\overline{a}_j \equiv \overline{a}'_j\,(\Theta_i)$ for each j, $(1 \le j \le n)$, then $a_{ji} = a'_{ji}$ for each j, $(1 \le j \le n)$. Thus $\mathbf{f}_i^{\mathcal{A}}(a_{1i}, \ldots, a_{ni}) = \mathbf{f}_i^{\mathcal{A}}(a'_{1i}, \ldots, a'_{ni})$. Therefore $\mathbf{f}^{\mathcal{P}}(\overline{a}_1, \ldots, \overline{a}_n) \equiv \mathbf{f}^{\mathcal{P}}(\overline{a}'_1, \ldots, \overline{a}'_n)\,(\Theta_i)$.

Hence Θ_i is a congruence relation. ∎

(b) Prove that $\mathcal{P}/\Theta_i \cong \mathcal{A}_i$ by the map $\overline{a}/\Theta_i \mapsto a_i$.

Proof: First of all, the map is well-defined because Θ_i depends only on the i-th coordinate; it is onto because all of $\mathcal{A}_i$ is used in the construction of a product; it is a homomorphism because Θ_i is a congruence relation; so the only thing that needs any checking at all is that it is one-to-one. But $a_i = a'_i$ implies $\overline{a} \equiv \overline{a}'\,(\Theta_i)$. ∎

(a),(b) **Solution:** A handful of you were smarter than me and observed (for (b)) that Θ_i is the kernel of the i-th projection map, and therefore (b) is immediate by the first isomorphism theorem. One of you was smarter than everyone else, and made the observation in (a), so of course Θ_i is trivially a congruence relation!

Therefore, if solved by the "best method", (a) and (b) are only one point questions.

(c) Prove that $\bigwedge_{i\in I} \Theta_i$ is "equality".

(That is, show that if $\overline{a} \equiv \overline{b}(\Theta_i)$ for all $i \in I$, then $\overline{a} = \overline{b}$.)

Proof: $\overline{a} \equiv \overline{b}(\Theta_i)$ for all $i \in I$ iff $a_i = b_i$ for all $i \in I$ iff $\overline{a} = \overline{b}$. ∎

(d) Prove that if $i \ne j \in I$, then $\Theta_i \vee \Theta_j = \iota$ (where ι is the "total" relation $\overline{a} \equiv \overline{b}(\iota)$ for all $\overline{a}$, $\overline{b}$.)

[Hint: Let $\overline{c}$ agree with $\overline{b}$ on all indices, except $c_i = a_i$, and then imitate the proof for a product of two groups given in class.)

Proof: Given any $\overline{a}$ and $\overline{b}$ in the product, define $\overline{c}$ by $c_k = b_k$ for $k \ne i$, and $c_i = a_i$. Then $\overline{c} \equiv \overline{b}(\Theta_j)$ since $j \ne i$, and $\overline{c} \equiv \overline{a}(\Theta_i)$ by the choice of c_i , Both Θ_i and Θ_j imply $\Theta_i \vee \Theta_j$ by definition of "$\vee$", so by transitivity $\overline{a} \equiv \overline{b}\,(\Theta_i \vee \Theta_j)$. ∎

[20] TOTAL

Remark: Question 2 Part (d) is not the most general statement. In fact a simple modification of the proof suggested for this part proves that if $i \ne j \in I$, then

$$\Theta_i \vee \bigwedge_{j\in I,\, j\ne i} \Theta_j = \iota$$

Proof: In fact, clearly $\overline{c} \equiv \overline{b}(\Theta_j)$ for *all* $j \ne i$, which is all we need. ∎