MATH 3322 Problem Set 8

March 21, 2019

Due: March 29, 2019

[4] **Question 1.** Find the Galois group of x^4+1 over $\mathbb{Q}$ and the Galois group of x^8-1 over $\mathbb{Q}$.

Source The next two exercises are from Dummit and Foote, §14.2.

Question 2. Let $\mathbb{F}$ be a field of characteristic different from 2.

[2] (a) If $K = \mathbb{F}(\sqrt{D_1}, \sqrt{D_2})$ where $D_1, D_2 \in \mathbb{F}$ are such that none of D_1, D_2, or D_1D_2 is a square in $\mathbb{F}$, prove that $K/\mathbb{F}$ is normal with $G(K/\mathbb{F})$ isomorphic to the Klein 4-group.

[3] (b) Conversely, if $K/\mathbb{F}$ is normal with $G(K/\mathbb{F})$ isomorphic to the Klein 4-group, then prove that $K = \mathbb{F}(\sqrt{D_1}, \sqrt{D_2})$ where $D_1, D_2 \in \mathbb{F}$ are such that none of D_1, D_2, or D_1D_2 is a square in $\mathbb{F}$.

Question 3. Let $f(x) = x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.
Note that if $g(x) = x^2 - 2x - 2$ then $f(x) = g(x^2)$.

(a) The following involves completely routine calculations and no marks are assigned. But it is always worthwhile to check the basic assumptions of the question!

Prove that $f(x)$ is irreducible over $\mathbb{Q}$, and verify that each of the following is a root of $f(x)$:

$$\alpha_1 = \sqrt{1+\sqrt{3}} \qquad \alpha_2 = \sqrt{1-\sqrt{3}} \qquad \alpha_3 = -\sqrt{1+\sqrt{3}} \qquad \alpha_4 = -\sqrt{1-\sqrt{3}}$$

Let $F = \mathbb{Q}(\sqrt{3})$, $K_1 = \mathbb{Q}(\alpha_1)$, $K_2 = \mathbb{Q}(\alpha_2)$, and $E = \mathbb{Q}(\alpha_1, \alpha_2)$.

[3] (b) Prove that $K_1 \neq K_2$ and that $K_1 \cap K_2 = F$.

[4] (c) Prove that each of K_1, K_2, and E is normal over F and that $G(E/F)$ is the Klein 4-group.

[4] (d) Prove that $\mathrm{Gal}(f(x) = 0 \,/\, \mathbb{Q})$ is isomorphic to the 8 element dihedral group D_4.

[20] TOTAL