MATH 3322 Problem Set 7

March 13, 2019

Due: March 20, 2019

Question 1. Let F be a field of characteristic p. [Recall then that in $F[x]$, $(a+b)^p = a^p + b^p$.] Define $\phi : F \to F : a \mapsto a^p$. ϕ is called the *Frobenius map.*

Remark: Ferdinand Georg Frobenius, 1849–1917.

[2] (a) Prove that ϕ is an endomorphism of F.

[1] (b) Prove that if F is finite, then ϕ is an automorphism and every element of F has a unique p-th root.

Question 2. Recall that every polynomial over a field of characteristic 0 is separable, that is, the irreducible factors have no repeated roots. Recall also that $f \in F[x]$ has a multiple root α iff α is a root of the formal derivative f'.

Suppose that F has finite characteristic p for some prime p.

[3] (a) Suppose that $f \in F[x]$ is irreducible with a multiple root.

Prove that $f = g(x^p)$ for some $g \in F[x]$.

[4] (b) Prove that every polynomial over F is separable iff every element of F has a p-th root.

[1] (c) Hence every polynomial over a finite field is separable.

Remark: So, e.g. if F is a field of characteristic p, then there are inseparable irreducible polynomials over the rational function field $F(t)$: $f(x) = x^p - t$ is one such.

[4] **Question 3.** We verified the values of the cyclotomic polynomials Φ_n for n prime and for small values of n ($n \le 5$), and listed all the values for $n \le 12$. Verify the values for $\Phi_6(x) = x^2 - x + 1$ and $\Phi_{12}(x) = x^4 - x^2 + 1$ by direct calculation.

Remark: Φ_{105} is the first cyclotomic polynomial in which a coefficient other than 0, 1, -1 occurs (-2 occurs twice). In general, large coefficients may appear in Φ_n where n is the product of many distinct prime factors. For instance, if $n = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, we find coefficients of ± 532.

Question 4. Prove that

[2] (a) If n is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$.

[3] (b) If $n = p^k$, p a prime, then $\Phi_n(x) = \Phi_p(x^{p^{k-1}})$.

[20] TOTAL