MATH 3322 Problem Set 6

March 7, 2019

Due: March 11, 2019

Question 1. By the primitive element theorem, Theorem 8, a primitive element for $K(\theta_1, \theta_2)$ can be found in the form $\theta_1 + k\theta_2$ for some $k \in K$. For instance, it is easy to see that $\sqrt{2} + i$ is a primitive element for $\mathbb{Q}(\sqrt{2}, i)$. For $(\sqrt{2} + i)^2 = 1 + 2i\sqrt{2}$, and so $\sqrt{2}i \in \mathbb{Q}(\sqrt{2} + i)$. But then $(\sqrt{2} + i)(\sqrt{2}i) = 2i - \sqrt{2} \in \mathbb{Q}(\sqrt{2} + i)$, from which it follows easily that both $i \in \mathbb{Q}(\sqrt{2} + i)$ and $\sqrt{2} \in \mathbb{Q}(\sqrt{2} + i)$.

Find a primitive element (with explanation) for each of

[2] (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;

[4] (b) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.

Question 2. Let K/F be a field extension, R a ring, and $F \leq R \leq K$.

[2] (a) Suppose that K/F is algebraic.

Prove that R is a field.

[2] (b) Give an example to show that the result fails when K/F is not algebraic.

Question 3. Let $[K : F] = n$. Prove the following:

[2] (a) For and $\alpha \in K$, $\phi_\alpha : K \to K : k \mapsto \alpha k$ is a linear transformation of the vector space ${}_F K$.

[2] (b) The map defined by sending each $\alpha \in K$ to the matrix of the linear transformation ϕ_α is an embedding of K as a subfield of the ring $\mathsf{M}_n(F)$ of $n \times n$ matrices over F.

(Hence every extension of F of degree n embeds as a subfield of $\mathsf{M}_n(F)$.

Question 4. Each of $p_2 = x^2 + x + 1$, $p_3 = x^3 + x + 1$, and $p_4 = x^4 + x + 1$ is irreducible over $\mathbb{F}_2$, the field with two elements. Let α be a root of p_2, let β be a root of p_3, and let γ be a root of p_4.

[3] (a) Find all the roots of p_2 in $E_2 = \mathbb{F}_2(\alpha)$, all the roots of p_3 in $E_3 = \mathbb{F}_2(\beta)$, and all the roots of p_4 in $E_4 = \mathbb{F}_2(\gamma)$.

[1] (b) Give a brief explanation of why E_2 does not embed in E_3 and of why E_3 does not embed in E_4.

[2] (c) Show that the map determined by $\alpha \mapsto \gamma^2 + \gamma + 1$ defines an embedding of $E_2 \hookrightarrow E_4$.

[20] TOTAL