MATH 3322 Problem Set 5

February 13, 2019

Due: March 01, 2019

[3] **Question 1.** Prove that if H is normal in G and both H and G/H are solvable, then so is G.

[2] **Question 2.** Prove that if H is normal in G and $H \leq K \leq G$, then $K/H = Z(G/H)$ iff $[K, G] \subseteq H$.

[2] **Question 3.** Prove that if A is a free abelian group on S and B is a free abelian group on T and $S \cap T = \emptyset$, then $A \oplus B$ is a free abelian group on $S \cup T$.

[3] **Question 4.** Prove (from the definition) that a homomorphic image of a nilpotent group is nilpotent.

Question 5. An abelian group A is called *divisible* if for every $a \in A$ and every $0 \neq n \in \mathbb{Z}$, there is $b \in A$ such that $nb = a$.

Clearly $\langle \mathbb{Q}; +, -, 0 \rangle$ is divisible.

[2] (a) Prove that a finite abelian group is not divisible.

Hint: Consider an element of prime power order.

Remark: A group G is *of bounded exponent* if for some n, $g^n = 1$ for all $n \in G$. Your proof will (likely) show in fact that an abelian group of bounded exponent is not divisible.

[3] (b) Prove that an (arbitrary) direct sum of abelian groups is divisible iff each of the summands is divisible.

[2] (c) Prove that a homomorphic image of a divisible abelian group is divisible.

[3] (d) Prove that $\mathbb{Q}/\mathbb{Z}$ is a divisible group in which every element is of finite order, and in which there are elements of any finite order.

[20] TOTAL