MATH 3322 Problem Set 4

February 7, 2019

Due: February 15, 2019

Notation: Let D_n be the dihedral group of symmetries of the regular n-gon, generated by two elements r and s such that $r^n = 1,\ s^2 = 1,\ rs = sr^{-1}$.

Note that many authors call this D_{2n}, to reflect the order of the group. Ames (our text) or Beachy and Blair use the name D_n as we have defined it; Dummit and Foote, another important reference, uses D_{2n}.

[3] **Question 1.** Let $p \in \mathbb{Z}$ be prime.
Prove that $\mathbb{Z}/p^n\mathbb{Z}$ has a unique composition series.

[4] **Question 2.** The sublattice diagrams for the quaternion group $\mathbf{Q}_8$ and the dihedral group $\mathbf{D}_4$ are attached.
Using only these diagrams and the fact that both groups have 8 elements, explain why $\mathbf{Q}_8$ has exactly 3 distinct composition series and $\mathbf{D}_8$ has exactly 7 distinct composition series.

[3] **Question 3.** Prove that if G has a composition series and $1 \neq H \trianglelefteq G$, then G has a composition series containing H.

[4] **Question 4.** Prove that D_{2^n} is solvable for each n.
Hint: Of course you only need to be able to find one subnormal series with abelian factors; the work you did on the previous question and the diagram for D_4 should give you a hint towards a fairly easy answer.

Question 5. Prove that:

[3] (a) Every subgroup of a solvable group is solvable.

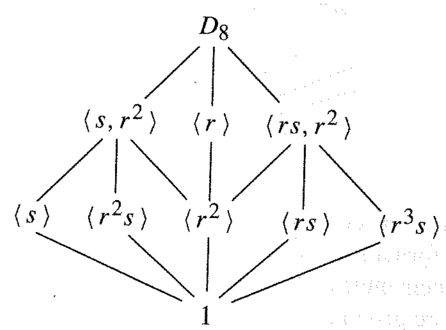
[3] (b) Every homomorphic image of a solvable group is solvable.

[20] TOTAL

From **Abstract Algebra** (Third Edition), David Dummit and Richard M. Foote,(Wiley 2004).

The first diagram shows what we call D_4.

(4) Using our usual notation for $D_8 = \langle r, s \rangle$, the lattice of D_8 is



(5) The lattice of subgroups of Q_8 is