# MATH 3322 Problem Set 2

January 23, 2019

## Due: January 30, 2019

[6] **Question 1.** A group $\langle G; *, ^{-1}, \mathbf{e} \rangle$ is *of exponent* $n$ if for all $g \in G$, $x^n = \mathbf{e}$. Equivalently, for an abelian group $A$, $na = 0$ for all $a \in A$.

$\mathbb{Z}_n$ is shorthand for the group of integers modulo $n$, more correctly written as $\mathbb{Z}/n\mathbb{Z}$. We take the elements of $\mathbb{Z}_n$ to be $\{0, 1, \ldots, n-1\}$.

[1] (a) Let $\langle A; +, -, 0 \rangle$ be an abelian group and $a \in A$ such that $na = 0$. Prove that there is a unique homomorphism $\varphi : \mathbb{Z}_n \to A$ such that $\varphi(1) = a$.

[5] (b) Let $X$ be any non-empty set. Consider the *weak direct power* $\mathbb{Z}_n^{(X)}$. [For details on weak direct powers and direct sums, see the next exercise.] Define $\varepsilon : X \to \mathbb{Z}_n^{(X)}$ by $\varepsilon(x)_x = 1$, $\varepsilon(x)_y = 0$ for $y \neq x$, $y \in X$.

Prove that $(\varepsilon, \mathbb{Z}_n^{(X)})$ is the free abelian group of exponent $n$ on $X$.

[8] **Question 2.** Let $(A_i)_{i \in I}$ be a non-empty family of abelian groups. Let

$$\bigoplus_{i \in I} A_i = \left\{ \overline{a} \in \prod_{i \in I} A_i : a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

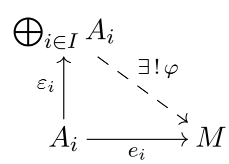
For each $i \in I$, let $\varepsilon_i : A_i \to \bigoplus_{i \in I} A_i$ be defined by

$$\varepsilon_i(a)_j = \begin{cases} a & j = i \\ 0 & j \neq i \end{cases}, \text{ for each } a \in A_i.$$

$(\bigoplus_{i \in I} A_i, (\varepsilon_i)_{i \in I})$ is called the *direct sum* of the abelian groups $(A_i)_{i \in I}$. If all the $A_i$ are equal to a single group $A$, we write it as $A^{(I)}$, and call it the *weak direct power* of $A$.

[2] (a) Prove that $\bigoplus_{i \in I} A_i$ is a subgroup of $\prod_{i \in I} A_i$, and that each $\varepsilon_i$ is a group homomorphism.

[6] (b) Prove that $\bigoplus_{i \in I} A_i$ satisfies the universal property described by the following diagram:




where $M$ is any abelian group and $e_i : A_i \to M$ is a family of group homomorphisms.

[For clarity: This means that we are given the diagram with all components $i$ simultaneously, and complete it by a single map $\varphi$.]

[6] **Question 3.** Let $(G_i)_{i\in I}$ be a non-empty family of groups. Let

$$\prod_{i\in I}^{W} G_i = \left\{ \overline{a} \in \prod_{i\in I} G_i : a_i = \mathbf{e}_i \text{ for all but finitely many } i \in I \right\}.$$

Let $\varepsilon_i : G_i \to \prod_{i\in I}{}^{W} G_i$ be defined by

$$\varepsilon_i(a)_j = \begin{cases} a & j = i \\ \mathbf{e}_j & j \neq i \end{cases}$$

Just as in Question 2, it is easy to see that $\prod_{i\in I}{}^{W} G_i$ is a subgroup of $\prod_{i\in I} G_i$, and that each $\varepsilon_i$ is an embedding. You can assume this result without proof.

[2] (a) Show that $\prod_{i\in I}^{W} G_i$ is a *normal* subgroup of $\prod_{i\in I} G_i$.

[1] (b) Show that for any group $G$ and any $g \in G$, the map $\psi : \mathbb{Z} \to G : z \mapsto g^z$ is a group homomorphism that sends $1 \in \mathbb{Z}$ to $g$.

[This shows that $\mathbb{Z}$ is not just the free abelian group on one generator, but in fact it is the free group on one generator.]

[3] (c) Show that if $G$ is a group with $a, b \in G$ such that $ab \neq ba$, then the pair of homomorphisms $e_1 : \mathbb{Z} \to z \mapsto a^z$, $e_2 : \mathbb{Z} \to z \mapsto b^z$, does not lift to a homomorphism $\mathbb{Z} \times \mathbb{Z} \to G$.

[This shows that the weak direct product of groups does not satisfy (for the class of all groups) the universal property described in 2(b), even for a product of only two groups! In particular, $\mathbb{Z}^2$ is *not* the free group on two generators.]

**Remark**: In abstract classes of algebras, the universal property 2(b) defines an object called the *free product*. Abelian groups (and more generally modules over a ring) are unusual in that the free product has a simple description by the direct sum. One of the basic properties of the free product is that if $\mathcal{F}_1$ is the free algebra on one generator in a variety $\mathcal{V}$, then the free product of $X$ copies of $\mathcal{F}_1$ is the free algebra on $X$.

[20] TOTAL