MATH 3322 Problem Set 1

January 14, 2019

Due: January 23, 2019

Note the extension by two days from the date published in the course outline.

Recall the following definitions and notation from the handout on Universal Algebra.

An *algebraic language* $\mathcal{L}$ is determined by a set $(\mathbf{f}_i)_{i \in I}$ of *function symbols*, each $\mathbf{f}_i$ a a $\nu(i)$-ary function symbol (where $\nu : I \to \omega \setminus \{ 0 \}$); and a set $(\mathbf{c}_k)_{k \in k}$ of *constant* symbols.

An *abstract algebra* $\mathcal{A}$ *for* $\mathcal{L}$ consists of a non-empty set A and actual operations and elements on A interpreting the symbols of $\mathcal{L}$.

An *assignment of values* in $\mathcal{A}$ is a map α : variables $\to A$.

You will need to review and refer to the definitions of *subalgebra*, *homomorphism*, and *congruence*, and Definition 0.4, how to evaluate a term in an algebra.

Here is the example from class:

Lemma 0.1 *Let* $\mathcal{B} \subseteq \mathcal{A}$ *be* $\mathcal{L}$*-algebras, and* $\mathbf{t}$ *an* $\mathcal{L}$*-term,* α *an assignment in* $\mathcal{A}$. *Then*

$$\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{t}^{\mathcal{A}}[\alpha]$$

Proof: If $\mathbf{t}$ is a variable v, then $\mathbf{t}^{\mathcal{B}}[\alpha] = \alpha(v) = \mathbf{t}^{\mathcal{A}}[\alpha]$.

If $\mathbf{t}$ is a constant symbol $\mathbf{c}$, then $\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{c}^{\mathcal{B}} = \mathbf{c}^{\mathcal{A}} = \mathbf{t}^{\mathcal{A}}[\alpha]$; the first and last equalities by the definition of evaluation, and the middle by the definition of subalgebra.

If $\mathbf{t}$ is a compound term $\mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ and the Lemma holds for $\mathbf{t}_1, \ldots, \mathbf{t}_n$, then

$$\mathbf{t}^{\mathcal{B}}[\alpha] = \mathbf{f}^{\mathcal{B}}(\mathbf{t}_1{}^{\mathcal{B}}[\alpha], \ldots, \mathbf{t}_n{}^{\mathcal{B}}[\alpha]) = \mathbf{f}^{\mathcal{A}}(\mathbf{t}_1{}^{\mathcal{A}}[\alpha], \ldots, \mathbf{t}_n{}^{\mathcal{A}}[\alpha]) = \mathbf{t}^{\mathcal{A}}[\alpha];$$

the first and last equalities by the definition of evaluations, and the middle one by the definition of subalgebra (for $\mathbf{f}$) and the assumption on the terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$. ∎

[4] **Question 1.** Let $\mathcal{B} \subseteq \mathcal{A}$ be $\mathcal{L}$-algebras.

(a) Prove that if an identity $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$, then the identity also holds in $\mathcal{B}$.

(b) Give a simple example in groups to show that the converse does not hold.

Question 2. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of $\mathcal{L}$-algebras, $\mathbf{t}$ an $\mathcal{L}$-term, and α an assignment in $\mathcal{A}$.

[3] (a) Prove that

$$\varphi(\mathbf{t}^{\mathcal{A}}[\alpha]) = \mathbf{t}^{\mathcal{B}}(\varphi \circ \alpha).$$

[3] (b) Prove that if φ is surjective, and if an identity $\mathbf{s} = \mathbf{t}$ holds in $\mathcal{A}$, then the identity also holds in $\mathcal{B}$.

[2] (c) Find a simple example in groups to show that the converse to (b) does not hold.

. . . continued

[8] **Question 3.** Let $(\mathcal{A}_i)_{i\in I}$ be a family of $\mathcal{L}$-algebras and $\mathcal{P} = \prod_{i\in I} \mathcal{A}_i$.

Define relations $(\Theta_i)_{i\in I}$ on $\mathcal{P}$ by $\overline{a} \equiv \overline{b}(\Theta_i)$ iff $a_i = b_i$. It's obvious that each Θ_i is an equivalence relation, and you don't have to prove this.

(a) Prove that each Θ_i is a congruence relation.

(b) Prove that $\mathcal{P}/\Theta_i \cong \mathcal{A}_i$ by the map $\overline{a}/\Theta_i \mapsto a_i$.

(c) Prove that $\bigwedge_{i\in I} \Theta_i$ is "equality".

(That is, show that if $\overline{a} \equiv \overline{b}(\Theta_i)$ for all $i \in I$, then $\overline{a} = \overline{b}$.)

(d) Prove that if $i \neq j \in I$, then $\Theta_i \vee \Theta_j = \iota$ (where ι is the "total" relation $\overline{a} \equiv \overline{b}(\iota)$ for all $\overline{a}$, $\overline{b}$.)

[Hint: Let $\overline{c}$ agree with $\overline{b}$ on all indices, except $c_i = a_i$, and then imitate the proof for a product of two groups given in class.)

[20] TOTAL

Remark: Question 2 Part (d) is not the most general statement. In fact a simple modification of the proof suggested for this part proves that if $i \neq j \in I$, then

$$\Theta_i \vee \bigwedge_{j\in I,\, j\neq i} \Theta_j = \iota$$