MATH 2170-19W Problem Set 6

March 15, 2019

Solutions

Remarks: Reduce an augmented matrix by column operations [and row operations]. At each stage, choose your "pivot" point as the coefficient entry with the least positive absolute value.

There may be more than one valid pathway to a solution, and I present only one.

You can always verify that a solution is correct by substituing the values obtained into the original equations, and computing.

I show the complete set-up initially, but then only display the parts of the augmented matrix that are being changed. At the end, I recover the unchanged parts in order to find the full solution.

Solutions in vector form are preferred but not mandatory. The same applies to "reduction of constants".

[4] **Question 1.** Find all positive integer solutions to

$$5x + 16y = 121$$

Solution:

$$\begin{array}{cc|cc}
5 & 16 & 121 & \qquad C_2 - 3C_1 \\
\hline
1 & 0 & x & \\
0 & 1 & y & \\
\hline
* & * & &
\end{array}$$

$$\begin{array}{cc|c}
5 & 1 & \qquad C_2 - 3C_1 \\
\hline
1 & -3 & \\
0 & 1 &
\end{array}$$

$$\begin{array}{cc|c}
0 & 1 & 121 \\
\hline
16 & -3 & x \\
-5 & 1 & y \\
\hline
u & v &
\end{array}$$

So we have new variables u and v, where $v = 121$ and we can take u as a parameter.

So $\begin{array}{rcl} 16u - 3(121) & = & x \\ -5u + 121 & = & y \end{array}$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -363 \\ 121 \end{bmatrix} + u \begin{bmatrix} 16 \\ -5 \end{bmatrix}$.

If we want to reduce the constants, note that $363/16 \approx 22$, and substitute $u = k + 22$ to get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 11 \end{bmatrix} + k \begin{bmatrix} 16 \\ -5 \end{bmatrix}.$$

It is certainly easier to find the positive solutions from the reduced form, but any method is acceptable. We want $x > 0$ and $y > 0$, so $-11 + 16k > 0$ or $k > 11/16$; and $11 - 5k > 0$ or $11/5 > k$. So $k = 1$ or $k = 2$; and the two positive solutions are $\langle x, y \rangle = \langle 5, 6 \rangle$ and $\langle x, y \rangle = \langle 21, 1 \rangle$.

[6] **Question 2.** Find all integer solutions to

$$3x + 7y + 11z = 157$$

Solution:

$$\begin{array}{ccc|cc} \textcircled{3} & 7 & 11 & 157 & \\ \hline 1 & 0 & 0 & x & C_2 - 2C_1 \\ 0 & 1 & 0 & y & C_3 - 3C_1 \\ 0 & 0 & 1 & z & \\ \hline * & * & * & & \end{array}$$

$$\begin{array}{ccc|c} 3 & \textcircled{1} & 2 & \\ \hline 1 & -2 & -3 & C_1 - 3C_2 \\ 0 & 1 & 0 & C_3 - 2C_1 \\ 0 & 0 & 1 & \\ \hline \end{array}$$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 157 \\ \hline 7 & -2 & 1 & x \\ -3 & 1 & -2 & y \\ 0 & 0 & 1 & z \\ \hline u & v & w & \end{array}$$

So we have new variables u, v, and w, with $v = 157$ and u and w as parameters.

$$\text{So} \left\{ \begin{array}{rcl} 7u - 2(157) + w & = & x \\ -3u + 157 - 2w & = & y \\ w & = & z \end{array} \right\}, \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -314 \\ 157 \\ 0 \end{bmatrix} + u \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

If we want to reduce the constants, note that $314/7 \approx 44$, and substitute $u = k + 44$ to get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 25 \\ 0 \end{bmatrix} + k \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Question 3. Consider the system of linear diophantine equations

$$\begin{cases} 10x & +6y & +3z & = & 232 \\ 9x & +7y & +6z & = & 278 \end{cases}$$

[8] (a) Find all the integer solutions.

[2] (b) Find all the solutions in positive integers.

Solution:

$$\begin{array}{rrr|c l} 10 & 6 & 3 & 232 & C_1 - 3C_3 \\ 9 & 7 & 6 & 278 & C_2 - 2C_3 \\ \hline 1 & 0 & 0 & x & \\ 0 & 1 & 0 & y & \\ 0 & 0 & 1 & z & \\ \hline * & * & * & & \end{array}$$

$$\begin{array}{rrr|c l} 1 & 0 & 3 & & C_3 - 3C_1 \\ -9 & -5 & 6 & & \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ -3 & -2 & 1 & & \\ \hline \end{array}$$

$$\begin{array}{rrr|c l} 1 & 0 & 0 & 232 & \\ -9 & -5 & 33 & 278 & R_2 + 9R_1 \\ \hline 1 & 0 & -3 & & \\ 0 & 1 & 0 & & \\ -3 & -2 & 10 & & \\ \hline \end{array}$$

$$\begin{array}{r|rr|c} 1 & 0 & 0 & 232 \\ \cline{2-4} 0 & -5 & 33 & 2366 \\ \hline 1 & 0 & -3 & \\ 0 & 1 & 0 & \\ -3 & -2 & 10 & \\ \hline \end{array}$$

$$\begin{array}{rr|c} -5 & 33 & C_2 + 6C_1 \\ \hline 0 & -3 & \\ 1 & 0 & \\ -2 & 10 & \\ \hline \end{array} \qquad \begin{array}{rr|c} -5 & 3 & C_1 + 2C_2 \\ \hline 0 & -3 & \\ 1 & 6 & \\ -2 & -2 & \\ \hline \end{array} \qquad \begin{array}{rr|c} 1 & 3 & C_2 - 3C_1 \\ \hline -6 & -3 & \\ 13 & 6 & \\ -6 & -2 & \\ \hline \end{array}$$

$$\begin{array}{rr|c} 1 & 0 & \\ \hline -6 & 15 & \\ 13 & -33 & \\ -6 & 16 & \\ \hline \end{array}$$

Now we reassemble the full matrix.

$$\begin{array}{rrr|r} 1 & 0 & 0 & 232 \\ 0 & 1 & 0 & 2366 \\ \hline 1 & -6 & 15 & x \\ 0 & 13 & -33 & y \\ -3 & -6 & 16 & z \\ \hline u & v & w & \end{array}$$

So we have three new variables $u = 232$, $v = 2366$, and w , which we take as a parameter, and get

$$\begin{aligned} x &= u - 6v + 15w \\ y &= 13v - 33w \\ z &= -3u - 6v + 16w \end{aligned}$$

or

$$\begin{aligned} x &= -13964 + 15w \\ y &= 30758 - 33w \\ z &= = -14892 + 16w \end{aligned}$$

So all solutions are given by set of equations. Notice that $30758/33 \approx 932$, so substitute $w = k + 932$ to get

$$\begin{aligned} x &= 16 + 15k \\ y &= 2 - 33k \\ z &= = 20 + 16k \end{aligned}$$

For positive solutions, from the equation for y , $k \leq 0$; and from the other two equations $-1 \leq k$, so the two positive solutions are $(x, y, z) = (16, 2, 20)$ and $(x, y, z) = (1, 35, 4)$.

In vector form, the general solution that we found is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 20 \end{bmatrix} + k \begin{bmatrix} 15 \\ -33 \\ 16 \end{bmatrix}$$