MATH 2170-19W Problem Set 3

February 7, 2019

Solutions

[6] **Question 1.** Given $a = 2^8 5^4 7^3 11^4 17^1$ and $b = 2^5 3^5 5^3 11^2 13^3$, find the prime power factorizations of $(a,\, b)$, $[a,\, b]$, and ab.

Solution: We had the formulas for these given in a theorem in class. The gcd is given by chosing the minimum exponent for each prime; the lcm is given by chosing the maximum exponent for each prime; and the product is given simply by taking the sums of the exponents for each prime. So:

$(a,\, b) = 2^5 5^3 11^2$,
$[a,\, b] = 2^8 3^5 5^4 7^3 11^4 13^3 17$, and
$ab = 2^{13} 3^5 5^7 7^3 11^6 13^3 17$.

[5] **Question 2.** Find all four pairs of numbers $\langle\, a,\, b\,\rangle$ with $0 < a \le b$, $(a,\, b) = 35$ and $[a,\, b] = 4900$.

Remark: This solution is provided with a lot of additional explanation. The marks are awarded primarily for finding the four pairs of numbers. A point will be taken off only if no explanation at all is provided.

Solution: $35 = 5^1 7^1$ and $4900 = 2^2 5^2 7^2$.

By the formulas for calculating gcd and lcm, the minimum exponent of each of 2, 3, 5 in a or b must be 0, 1, 1 respectively, and the maximum value must be 2, 2 2 respectively. This apparently gives us 8 choices for a and b (two choices for each of the three exponents) but the requirement that $a < b$ reduces this to four pairs:

powers of	2	5	7		2	5	7	
	0	1	1	35	2	2	2	4900
	0	1	2	245	2	2	1	700
	0	2	1	175	2	1	2	980
	0	2	2	1225	2	1	1	140

the next 4 rows repeat the ones above, in reverse order

giving us the four pairs $\langle\, 35,\, 4900\,\rangle$, $\langle\, 245,\, 700\,\rangle$, $\langle\, 175,\, 980\,\rangle$, and $\langle\, 140,\, 1225\,\rangle$.

[4] **Question 3.** Show that every positive integer can be written uniquely in the form $n = ab$, where a is square free and b is a square.

Proof: 1 is trivially both square and square free.

So suppose $n > 1$ and has prime power factorization

$$n = {p_1}^{k_1} \times \cdots \times {p_t}^{k_t} ,$$

the p_i being distinct primes actually occurring in the factorization of n. Recall that n is a perfect square iff all the exponents are even, and n is square-free if it has no square divisors, hence iff all the exponents are 1. For each i, write $k_i = 2q_i + r_i$, $r_i = 0$ or 1. Then $n = \left(\prod_{i=1}^{t} p_i^{q_i}\right)^2 \times \left(\prod_{i=1}^{t} p_i^{r_i}\right)$, clearly a product of a perfect square and a square-free integer.

Note: There several are different ways of presenting this solution.

One could also write $n = \left(\prod_{i=1}^{t} p_i^{q_i}\right)^2 \times \left(\prod_{i=1,(r_i=1)}^{t} p_i\right)$, for instance. ∎

[5] **Question 4.** Observe that $(4m+1)(4n+1) = 4(4mn+m+n)+1$ and that $4m+3)(4n+3) = 4(4mn+3m+3n+2)+1$.

Note: There was a misplaced parenthesis on the right hand side of the first equation.

Prove that every positive integer n of the form $4k+3$ has an odd number of prime factors of the form $4k+3$.

Proof: Suppose that n has an even number $2t$ of prime factors of the form $4k+3$. Then group them in pairs $(p_1 p_{t+1})(p_2 p_{t+2}) \cdot \ldots \cdot (p_t p_{2t})$. By the second formula given, each one of these products is of the form $4k+1$, and by the first formula given, any product of numbers of the form $4k+1$ also has that form. So if n has the form $4k+3$, it must have an odd number of prime factors of the form $4k+3$. ∎

Note: Clearly this same solution can be presented in a more elegant way using congruences mod 4, from the next chapter.

[20] TOTAL