MATH 2170-19W Problem Set 2

February 1, 2019

Solutions

[8] **Question 1.** Using the Euclidean algorithm,

(a) determine $d = (112860,\ 51876)$;

(b) find integers x and y such that $d = 112860x + 51876y$;

(c) find $m = [112860,\ 51876]$.

Solution: We learned the following formulas: $r_{-1} = a$, $r_0 = b$, q_{i+1} is determined by division: $r_{i-1} = r_i q_{i+1} + r_{i+1}$, $0 \leq r_{i+1} < r_i$, so we get rules as follows (with $x_{-1} = 1$, $y_{-1} = 0$, $x_0 = 0$, and $y_0 = 1$):

$$\begin{cases} r_k &= r_{k-2} - q_k r_{k-1} \\ x_k &= x_{k-2} - q_k x_{k-1} \\ y_k &= y_{k-2} - q_k y_{k-1} \end{cases}$$

Therefore in this case:

q_{i+1}	r_i	x_i	y_i
	112860	1	0
2	51876	0	1
5	9108	1	-2
1	6336	-5	11
2	2772	6	-13
3	792	-17	37
2	396	57	-124
	0		

Therefore $d = 396 = 57 \times 112860 - 124 \times 51876$ and $m = \frac{112860 \times 51876}{396} = 285 \times 51876 = 14784660$.

Note On the test and the final exam, you will be expected to work out any similar problems according to the algorithm learned in class and given in the solution here. On this assignment, you earned full marks for a correct final answer regardless of the method used. Note that the explicit algorithm gives you the required values of x and y much more quickly than the long-hand method.

[8] **Question 2.** You can take it as "given" that $(30,\, 42,\, 70) = 2$. Recall that $(a, b, c) = ((a, b), c) = (a, (b, c))$.

Find integers x, y, and z such that $30x + 42y + 70z = 2$.

Comment: There are several different pathways through to the solution, and they lead to different numerical answers. All correct solutions are acceptable.

Solution: $(30,\, 42,\, 70) = ((70,\, 42),\, 30) = (d,\, 30)$

q_{i+1}	r_i	x_i	y_i
	70	1	0
1	42	0	1
1	28	1	-1
2	14	-1	2
	0		
$d = 14 = -1 \times 70 + 2 \times 42$			

q_{i+1}	r_i	x_i	y_i
	30	1	0
2	14	0	1
1	2	1	-2
	0		
$(30,\, d) = 2 = 1 \times 30 - 2 \times d$			

Therefore $2 = 1 \times 30 - 2 \times d = 1 \times 30 - 2 \times (-1 \times 70 + 2 \times 42) = 1 \times 30 - 4 \times 42 + 2 \times 70$.

Note I displayed a full formal solution here, but especially in the second step, it would be entirely legitimate to say "By inspection, it is obvious that we can write $2 = 30 - 2 \times d$."

[4] **Question 3.** Prove that the pair of equations

$$\begin{aligned} (a,\, b) &= d \\ ab &= m \end{aligned}$$

has a simultaneous solution iff $d^2 \mid m$.

Proof: Suppose that a and b satisfy this pair of equations. Then $a = dx$ and $b = dy$ for some x and y, so $m = ab = dxdy = d^2xy$, that is, $d^2|m$.

On the other hand if $d^2|m$ then $m = d^2z = d(dz)$ for some z. Clearly $(d,\, dz) = d$, so $a = d$, $b = dz$ is a solution. ∎

Note These aren't the only correct arguments to give, but probably the most natural. You get your marks for any correct proof, not just for the one I had in mind.

A common error was to do only one direction of the "if and only if".

The converse direction is not immediately obvious or "mechanical", and there were a few partial solutions.

[20] TOTAL