MATH 2170-19W Problem Set 1

January 24, 2019

SOLUTIONS

[3] **Question 1.** Prove:

If $a \mid c$ and $b \mid d$, then $ab \mid cd$.

Proof: If $a \mid c$ and $b \mid d$, then for some m and n, $am = c$ and $bn = d$. Then $cd = (am)(bn) = (ab)(mn)$, so $ab \mid cd$. ∎

[8] **Question 2.** An integer n is *even* if $2 \mid n$, and *odd* otherwise.

Prove:

[2] (a) n is even iff $n = 2m$ for some integer m, and n is odd iff $n = 2k+1$ for some integer k.

Proof: If n is an integer, then $n = 2q + r$, $0 \le r < 2$. So $r = 0$ or $r = 1$. n is even iff $2 \mid n$ iff $r = 0$, so n is odd iff $n = 2q + 1$. ∎

[1] (b) n is odd iff $n+1$ is even.

Proof: n is odd iff $n = 2m+$ for some m iff $n+1 = 2m+1+1 = 2(m+1)$ iff $n is even,$. ∎

[2] (c) $n(n+1)$ is even.

Proof: By the preceding part, one of n, $n+1$ is even, that is, divisible by 2, so $n(n+1)$ is divisible by 2. ∎

[3] (d) If n is odd, then $n^2 - 1$ is divisible by 8.

Proof: If n is odd, then $n = 2m+1$ for some m, and so

$$n^2 - 1 = (2m+1)^2 - 1 = 4m^2 + 4m + 1 - 1 = 4m(m+1).$$

By the preceding part, $m(m+1)$ is divisible by 2, so $n^2 - 1 = 4m(m+1)$ id divisible by 8. ∎

[6] **Question 3.**

[2] (a) Prove that the remainder upon dividing $ax + b$ by a is the same as the remainder upon dividing b by a.

Proof: $b = qa + r$, $0 \le r < a$, so $ax + b = ax + qa + r = a(x+q) + r$, so by the uniqueness of quotient and remainder, r is the remainder on dividing $ax + b$ by a. ∎

[3] (b) If we divide an integer n by 5, the possible remainders are 0, 1, 2, 3, 4. What are the possible remainders when n^2 is divided by 5?

[Hint: Set $n = 5k + r$, $0 \le r < 5$.]

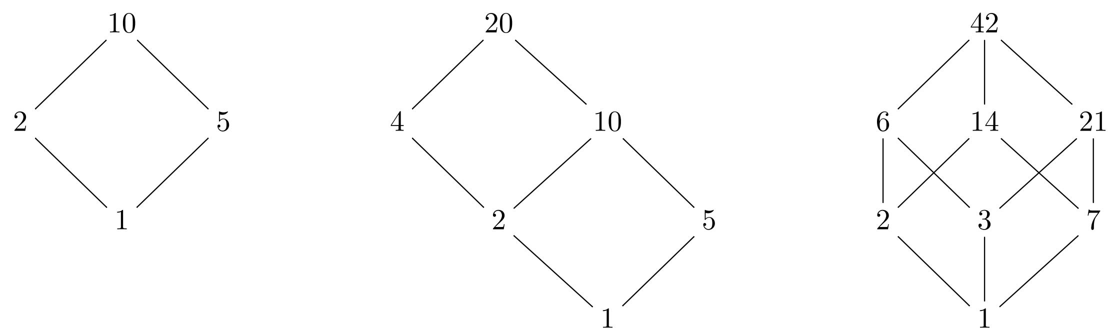
Solution: If $n = 5k + r$, $0 \le r < 5$, then $n^2 = 25k^2 + 10k + r^2 = 5(5k^2 + 2k) + r^2$, so by the preceding part the remainder on dividing n^2 by 5 is the same as the remainder on dividing r^2 by 5. But $0^2 = 1$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9 = 5 \times 1 + 4$, and $4^2 = 16 = 5 \times 3 + 1$. So the remainder on dividing n^2 by 5 is 0, 1, or 4.

[1] (c) Can an integer m ending in 3 be the square of an integer? (explain).

Solution: You can solve this in the same style as part (b), by considering the squares of the possible last digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 .

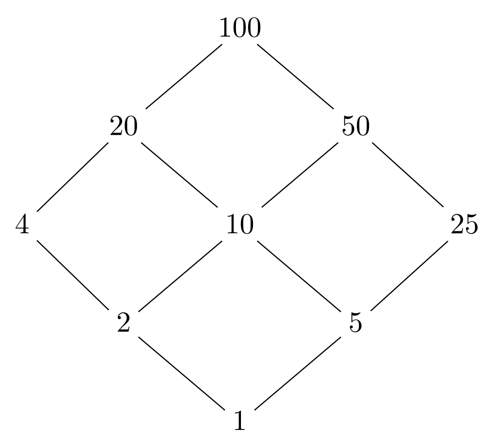
But it is much faster to use part (b): If $m = 10k + 3$, then $m = 5(2k) + 3$, so by part (b) m is not a square.

[3] **Question 4.** Here are diagrams of all the positive divisors of $n = 10$, $n = 20$, and $n = 42$ respectively, drawn as in class. The divisors are arranged according to the divisibility relations: there is a line between two integers if the lower number divides the upper number, and no other divisor fits in between. Furthermore, in a horizontal row, we list the divisors in increasing order.



Draw a similar diagram showing the relations between all the positive divisors of 100 . $(100 = 2^2 5^2)$

Solution: From the hint we easily find the list of all positive divisors of 100 : 1, 2, 4, 5, 10, 20, 25, 50, 100.



[20] TOTAL