MATH 2170

Introduction to Number Theory

Arithmetic Functions: Supplementary Notes.

March, 2019

Definition An *arithmetic function* is a function $f : \mathbb{Z}^+ \to \mathbb{C}$, where $\mathbb{Z}^+$ is the set of positive integers. All the arithmetic functions that we care about in this course will have range included in $\mathbb{Z}$.

Some interesting and useful arithmetic functions.

Let $n > 0$:

- $\underline{0}(n) = 0$, $\quad \mathbb{1}(n) = 1$, $\quad \mathrm{id}(n) = n$.
- $\underline{1}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$.
- $d(n)$ is the number of positive divisors of n.
- $\sigma(n)$ is the sum of the positive divisors of n.
- $\sigma_k(n)$ is the sum of the k-th powers of the positive divisors of n.
- $\omega(n)$ is the number of distinct primes that divide n.
- $\Omega(n)$ is the sum of the exponents in the prime power factorization of n.
- $\phi(n)$ is Euler's function: the number of elements in a reduced residue system modulo n.
- $\mu(n)$ is the *Möbius function*: $\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is square-free} \\ 0 & \text{otherwise} \end{cases}$.

Definition Note that this definition does NOT occur in Sections 4.2 and 4.3.

The convolution product: If f and g are arithmetic functions, then $f * g$ is the function defined by the rule

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Equivalently, we have $f * g(n) = \sum_{d_1 d_2 = n} f(d_1) g(d_2)$.

If we take $g = \mathbb{1}$, we see that $f * g(n) = \sum_{d|n} f(d)$. This is the particular case of the operation $*$ that is discussed in Sections 4.2 and 4.3; the main theorem about this special case, Theorem 4.4, generalizes to $*$ with virtually no change in the proof, as we shall see in class.

Some properties of the convolution

(i) $f * g = g * f$

(ii) $f * (g * h) = (f * g) * h$

Proof:

$(f * (g * h)(n)) = \sum_{d_1 d_2 = n} f(d_1)(g * h)(d_2) = \sum_{d_1 d_2 = n} f(d_1) \left(\sum_{d_3 d_4 = d_2} g(d_3) h(d_4) \right) = \sum_{d_1 d_3 d_4 = n} f(d_1) g(d_3) h(d_4)$, and then we simplify grouping to the left instead of to the right to prove the result.

(iii) If we define the function $g + h$ by the rule $(g + h)(n) = g(n) + h(n)$, then $f * (g + h) = f * g + f * h$.

(iv) $f * \underline{0} = \underline{0}$, $\quad f * \underline{1} = f$.

(v) $f * \mathbb{1} = \sum_{d|n} f(d)$, so $d(n) = \mathbb{1} * \mathbb{1}$ and $\sigma = \mathrm{id} * \mathbb{1}$.

Definition An arithmetical function $f \neq \underline{0}$ is called *multiplicative* if whenever $(m, n) = 1$ then $f(mn) = f(m)f(n)$.

f is called *totally multiplicative* if $f(mn) = f(m)f(n)$ for all m and n.

Facts: Every totally multiplicative function is multiplicative.

$\underline{1}$, $\mathbb{1}$, and id are easily seen to be totally multiplicative. ϕ is multiplicative (Theorem 2.19). It follows from property (v) above and Theorem 4.6* below that d and σ are multiplicative.

Ω and ω are examples of *totally additive* and *additive* functions respectively: arithmetic functions f that satisfy $f(mn) = f(m) + f(n)$ for all m, n, or for all m, n such that $(m, n) = 1$ respectively.

It follows from the definition of μ and the laws of exponents that since ω is additive, μ is multiplicative.

We will prove the following version of Theorem 4.4:

Theorem 4.4*.

If f and g are multiplicative, then so is $f * g$.

We will then state and prove three important results in the text.

In the notation of convolutions, these are:

Theorem 4.6 $\phi * \mathbb{1} = \mathrm{id}$.

Theorem 4.7 $\mu * \mathbb{1} = \underline{1}$.

Theorem 4.8 If $F = f * \mathbb{1}$ then $f = \mu * F$.