MATH 1510 Tutorial Worksheet 8

November 1, 2007

SOLUTIONS

Question 1. Given the following information about $g(x)$, sketch the graph of $y = g(x)$:

g has no symmetry and is not defined at $x = 0$. $g(-1) = -1$, $g(1) = g(3) = 0$, and $g(2) = -1$. $\lim_{x\to-\infty} g(x) = 0$, $\lim_{x\to\infty} g(x) = 1$, $\lim_{x\to 0^-} g(x) = -\infty$, and $\lim_{x\to 0^+} g(x) = \infty$.

$g'(x) = 0$ only when $x = 2$, and $g'(x)$ is negative when $x < 0$ or $0 < x < 2$. Otherwise $g'(x)$ is positive.

$g''(x) = 0$ only when $x = 3$, and $g''(x)$ is negative when $x < 0$ or when $x > 3$. Otherwise, $g''(x)$ is positive.

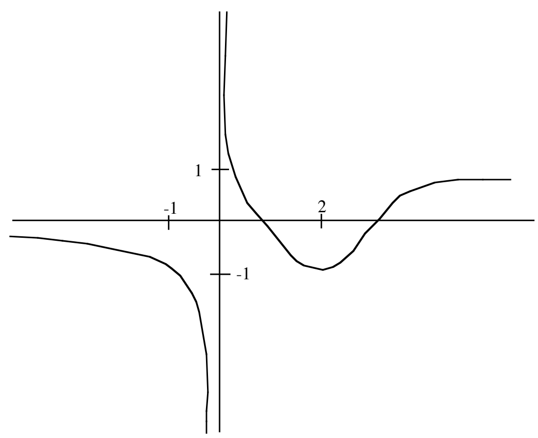
Solution:

1. In addition to what is stated explicitly, we see that g has a vertical asymptote at $x = 0$ and that it has two horizontal asymptotes: $y = 1$ in the positive direction and $y = 0$ in the negative direction.

2. g is decreasing on $(-\infty, 0)$ and on $(0, 2]$; g is increasing on $[2, \infty)$. g has a local minimum at $(2, -1)$ and this is the only local extreme point.

3. g is concave down on $(-\infty, 0)$ and on $[3, \infty)$; g is concave up on $(0, 3]$. There is only one inflection point, at $(3, 0)$.

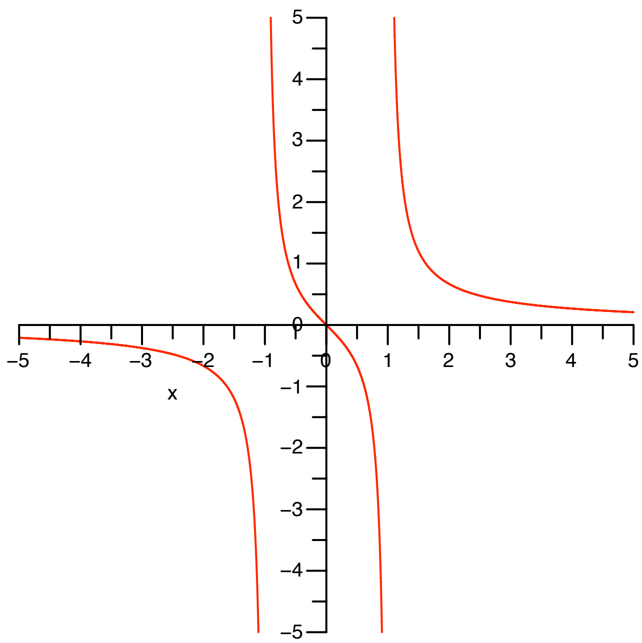
4.



Question 2. Sketch the graph of:

(a) $f(x) = \dfrac{x}{x^2-1}$

1. $f(0)=0$, and this is the only intercept. f is not defined at $x=\pm 1$.
   $\lim_{x\to -1^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = +\infty$, $\lim_{x\to 1^-} f(x) = -\infty$, and $\lim_{x\to 1^+} f(x) = +\infty$. (Just consider in each case the sign of each factor x, $x-1$, and $x+1$, near the limit point and on the appropriate side). Therfore f has odd vertical asymptotes at $x=-1$ and $x=1$.
   $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1-\frac{1}{x}^2} = 0$, (and similarly for $x \to -\infty$), so the x-axis is a horizontal asymptote to f.
2. $f'(x) = -\frac{x^2+1}{(x^2-1)^2}$, so $f'(x)$ is never equal to zero, and is negative everywhere it is defined. Therefore f is decreasing on each of the three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. **NOTE** that it would be wrong to lump all three intervals together, as f definitely does not decrease from one interval to the next. The endpoints are *not* included because they are not in the domain of f. There are no critical points and no local extreme points
3. $f''(x) = 2\frac{x(x^2+3)}{(x^2-1)^3}$. There is one value at which $f''(x)=0$, namely $x=0$, but f'' can also change sign across the points where f is not defined. In fact the denominator is negative for $-1<x<1$ and positive outside that interval, so in fact the sign of f'' changes three times: f is concave down on $(-\infty,-1)$ and on $[0,1)$; and concave up on $(-1,0]$ and on $(1,\infty)$. There is an inflection point at $(0,0)$. (Note that $f'(0)=-1$, the slope of the tangent line at the inflection point).
4.



(b) $k(x) = (x^2 - 3x + 2)\mathbf{e}^x$.

You may use the following fact: If $p(x)$ is any polynomial, then $\lim_{x\to-\infty} p(x)\mathbf{e}^x = 0$.

Solution: $k(x)$ is clearly defined everywhere

1. $k(x)$ is clearly defined everywhere, and we are given that $\lim_{x\to-\infty} k(x) = 0$. There are no vertical asymptotes, and the negative x-axis is a horizontal asymptote. There is no symmetry. $k(0) = 2$, the y-intercept, and $k(x) = 0$ if $x^2 - 3x + 2 = 0$, that is, if $x = 1$ or $x = 2$: the x-intercepts.

2. $k'(x) = (x^2 - x - 1)\mathbf{e}^x$. The critical points occur when $x^2 - x - 1 = 0$, that is, when $x = \dfrac{1 \pm \sqrt{5}}{2}$. Since the graph of $x^2 - x - 1$ is a parabola opening upwards and $\mathbf{e}^x$ is always positive, k' is positive for $x < \frac{1-\sqrt{5}}{2}$ and for $x > \frac{1+\sqrt{5}}{2}$, and negative for $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$. Therefore $k(x)$ is increasing on $\left(-\infty, \frac{1-\sqrt{5}}{2}\right]$ and on $\left[\frac{1+\sqrt{5}}{2}, \infty\right)$; and decreasing on $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$.

   $k(x)$ has a local maximum at $x = \frac{1-\sqrt{5}}{2}$ and a local minimum at $x = \frac{1+\sqrt{5}}{2}$. [We need a calculator to approximate the function values: these two points are roughly $(-.6,\ 2.3)$ and $(1.6,\ -1.2)$.

3. $k''(x) = (x^2 + x - 2)\mathbf{e}^x$. Potential inflection points occur when $x^2 + x - 2 = 0$, that is when $x = -2$ or $x = 1$. Again, the graph of $x^2 + x - 2$ is a parabola opening upwards and $\mathbf{e}^x$ is always positive, so $k''(x)$ is positive for $x < -2$ and for $x > 1$; $k''(x)$ is negative for $-2 < x < 1$. Therefore $k(x)$ is concave up on $(-\infty, -2]$ and on $[1, \infty)$; and $k(x)$ is concave down on $[-2, 1]$. There are inflection points when $x = -2$ and when $x = 1$; the corresponding function values are approximately 1.6 and 0 .

4.