MATH 1510 Tutorial Worksheet 6

October 18, 2007

SOLUTIONS

Question 1. Find $f'(t)$ if:

(a) $f(t) = \log_{10}(10t + \sqrt{t^2 + 100})$

$$f'(t) = \frac{1}{(10t + \sqrt{t^2+100}) \ln 10} \left(10 + \frac{2t}{2\sqrt{t^2+100}} \right)$$

(b) $f(t) = \mathbf{e}^{t^3-1}(t^4 + 3t)$

$$f'(t) = \mathbf{e}^{t^3-1}(3t^2)(t^4+3t) + \mathbf{e}^{t^3-1}(4t^3+3)$$

(c) $f(t) = \dfrac{\mathbf{e}^t - \ln(t)}{t}$

$$f'(t) = \frac{(\mathbf{e}^t - 1/t)t - (\mathbf{e}^t - \ln(t)) \cdot 1}{t^2}$$

Question 2.

(a) Find $\dfrac{dy}{dx}$ if $\ln(x^2 + y^2) = xy$.

(Implicit differentiation. y is treated as a function of x.)

$$\begin{aligned} \ln(x^2+y^2) &= xy \\ \frac{1}{x^2+y^2}\left(2x + 2y\frac{dy}{dx}\right) &= 1 \cdot y + x\frac{dy}{dx} \\ \left(\frac{2y}{x^2+y^2} - x\right)\frac{dy}{dx} &= y - \frac{2x}{x^2+y^2} \\ \frac{dy}{dx} &= \frac{y - \frac{2x}{x^2+y^2}}{\frac{2y}{x^2+y^2} - x} \end{aligned}$$

Comment: There is nothing wrong with writing y' for dy/dx; this is a matter of personal preference. The simplified answer looks a little nicer:

$$\frac{dy}{dx} = \frac{2y - x^3 - xy^2}{x^2y + y^3 - 2x}$$

(b) Find the value of $\dfrac{d^2y}{(dx)^2}$ at the point $\langle \mathbf{e}, 1 \rangle$ if $x\mathbf{e}^y - y = x^2 - 1$.

(Implicit differentiation. y is treated as a function of x.)

$$\begin{aligned} x\mathbf{e}^y - y &= x^2 - 1 \\ 1 \cdot \mathbf{e}^y + x\mathbf{e}^y y' - y' &= 2x \\ (x\mathbf{e}^y - 1)y' &= 2x - \mathbf{e}^y \end{aligned}$$

When $x = \mathbf{e}$ and $y = 1$, $y' = \dfrac{\mathbf{e}}{\mathbf{e}^2 - 1}$.

Differentiate again to find y'':

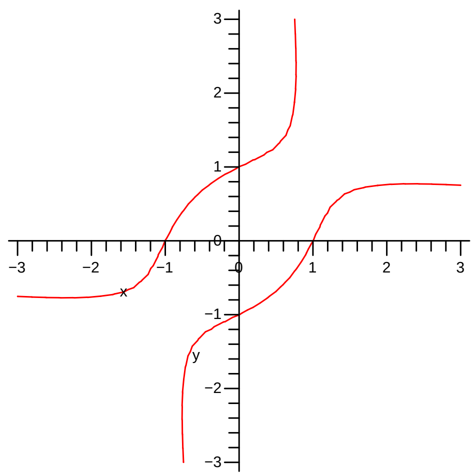
$$\begin{aligned}
(x\mathbf{e}^y - 1)y' &= 2x - \mathbf{e}^y \\
(1 \cdot \mathbf{e}^y - x\mathbf{e}^y y')y' - (x\mathbf{e}^y - 1)y'' &= 2 - \mathbf{e}^y y'
\end{aligned}$$

Put in the known values of x, y, and y':

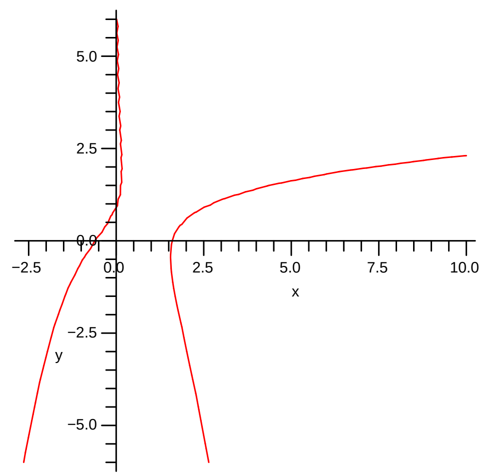
$$\begin{aligned}
(1 \cdot \mathbf{e}^y - x\mathbf{e}^y y')y' - (x\mathbf{e}^y - 1)y'' &= 2 - \mathbf{e}^y y' \\
\left(\mathbf{e}^1 - \mathbf{e} \cdot \mathbf{e}^1 \frac{\mathbf{e}}{\mathbf{e}^2 - 1}\right) \frac{\mathbf{e}}{\mathbf{e}^2 - 1} - (\mathbf{e} \cdot \mathbf{e}^1 - 1)y'' &= 2 - \mathbf{e}^1 \frac{\mathbf{e}}{\mathbf{e}^2 - 1} \\
\frac{-\mathbf{e}^2}{\left(\mathbf{e}^2 - 1\right)^2} - \left(\mathbf{e}^2 - 1\right) y'' &= 2 - \frac{\mathbf{e}^2}{\mathbf{e}^2 - 1} \\
-\mathbf{e}^2 - \left(\mathbf{e}^2 - 1\right)^3 y'' &= 2\left(\mathbf{e}^2 - 1\right)^2 - \mathbf{e}^2\left(\mathbf{e}^2 - 1\right) \\
-\left(\mathbf{e}^2 - 1\right)^3 y'' &= 2\left(\mathbf{e}^4 - 2\mathbf{e}^2 + 1\right) - \left(\mathbf{e}^4 - \mathbf{e}^2\right) + \mathbf{e}^2 \\
y'' &= -\frac{\mathbf{e}^4 - 2\mathbf{e}^2 + 2}{\left(\mathbf{e}^2 - 1\right)^3}
\end{aligned}$$

(Note that $y'' = -\left(\mathbf{e}^2 - 1\right)^{-1} - \left(\mathbf{e}^2 - 1\right)^{-3}$.)

For your information only (none of this is required for the solution of the question), here are sketches of the functions in Question 2(a) and 2(b):



$\ln(x^2 + y^2) = xy$



$x\mathbf{e}^y - y = x^2 - 1$

Question 3. Find $f'(x)$ if

(a) $f(x) = \left(1 + x^2\right)^{1/x^2}$

(logarithmic differentiation)

$$\begin{aligned}
f(x) &= \left(1+x^2\right)^{1/x^2} \\
\ln\left(f(x)\right) &= \frac{1}{x^2}\ln\left(1+x^2\right) \\
\frac{f'(x)}{f(x)} &= \frac{-2}{x^3}\ln\left(1+x^2\right) + \frac{1}{x^2}\cdot\frac{1}{1+x^2}\cdot 2x \\
f'(x) &= \left(1+x^2\right)^{(1/x^2)}\left[\frac{-2\ln\left(1+x^2\right)}{x^3} + \frac{2}{x\left(1+x^2\right)}\right]
\end{aligned}$$

(b) $f(x) = \dfrac{(x^2+2)\sin(x+1)\mathbf{e}^{x^3}}{x^3\cos(x)}$

(logarithmic differentiation. Note that differentiation by several uses of the product and quotient rule also works, but is much, much more work and not at all the answer you should give!)

$$\begin{aligned}
f(x) &= \frac{(x^2+2)\sin(x+1)\mathbf{e}^{x^3}}{x^3\cos(x)} \\
\ln\left(f(x)\right) &= \ln\left(x^2+2\right) + \ln(\sin(x+1)) + x^3 - 3\ln(x) - \ln(\cos(x)) \\
\frac{f'(x)}{f(x)} &= \frac{2x}{x^2+2} + \frac{\cos(x+1)}{\sin(x+1)} + 3x^2 - \frac{3}{x} - \frac{-\sin(x)}{\cos(x)} \\
f'(x) &= \left(\frac{(x^2+2)\sin(x+1)\mathbf{e}^{x^3}}{x^3\cos(x)}\right)\left[\frac{2x}{x^2+2} + \cot(x+1) + 3x^2 - \frac{3}{x} + \tan(x)\right]
\end{aligned}$$

Comment: Simplification isn't necessary, but it helps in writing down a long answer!

Question 4. A question for thought and exploration. Note that we would *NEVER* ask a question like this on a test or exam!

Consider the function $f(x) = \mathbf{e}^x(a\cos(x) + b\sin(x))$.
Find $f'(x)$.

Solution:

$$\begin{aligned}
f'(x) &= \mathbf{e}^x(a\cos(x) + b\sin(x)) + \mathbf{e}^x(a(-\sin(x)) + b\cos(x)) \\
&= \mathbf{e}^x((a+b)a\cos(x) + (b-a)\sin(x))
\end{aligned}$$

You should be able to see how to use the expression for $f'(x)$ to work out $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$ "mechanically", that is, without doing any more differentiation.

Solution: $f'(x)$ has exactly the same pattern as $f(x)$, just with different values of the coefficients of $\cos(x)$ and $\sin(x)$. For instance, if $a = 1$ and $b = 2$ then $f(x) = \mathbf{e}^x(\cos(x) + 2\sin(x))$, so $f'(x) = \mathbf{e}^x(3\cos(x) + 1\sin(x))$, and therefore $f''(x) = \mathbf{e}^x(4\cos(x) - 2\sin(x))$.

All of these follow from the formula we developed for $f'(x)$.
Therefore in general we find:

$$\begin{aligned} f(x) &= \mathbf{e}^x(a\cos(x) + b\sin(x)) \\ f(x) &= \mathbf{e}^x((a+b)a\cos(x) + (b-a)\sin(x)) \\ f(x) &= \mathbf{e}^x(2b\cos(x) - 2a\sin(x)) \\ f(x) &= \mathbf{e}^x(2(b-a)\cos(x) - 2(a+b)\sin(x)) \\ f(x) &= \mathbf{e}^x(-4a\cos(x) - 4b\sin(x)) \end{aligned}$$

Can you see the pattern developing? Can you guess a general formula for $f^{(n)}(x)$? (Actually, you would have to set this up as a family of different formulas. The pattern for $f^{(4n)}(x)$ is the easiest one to figure out.)
Solution: Most of this we leave for you for further thought. But clearly we see the following pattern developing:

$$f^{(4n)}(x) = (-4)^n\mathbf{e}^x(a\cos(x) + b\sin(x))$$

Work out $f^{(4n+1)}(x)$, $f^{(4n+2)}(x)$, $f^{(4n+3)}(x)$, for yourself!
[You can find such a formula in many references—maybe even in your textbook!]