MATH 1510 Tutorial Worksheet 4

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SOLUTIONS

Question 1. Find $f'(x)$ if:

(a) $f(x) = 5\sqrt[5]{x} + 7\sqrt[7]{x} + 9\sqrt[9]{\pi}$

Solution: Note that $\sqrt[n]{x} = x^{1/n}$, so $\dfrac{d}{dx}\sqrt[n]{x} = \dfrac{1}{n}x^{(1/n)-1} = \dfrac{1}{n}x^{(1-n)/n}$.

Therefore $f'(x) = 5\left(\dfrac{1}{5}\right)x^{-4/5} + 7\left(\dfrac{1}{7}\right)x^{-6/7} + 0$ ∎

(b) $f(x) = (x^5+1)(x^6+2)(x^7+3)$

Solution: (This can be solved directly in one line if you want, as the derivative of a product of three factors. We show the more detailed two-line solution.)

$$\begin{aligned} f'(x) &= 5x^4(x^6+2)(x^7+3) + (x^5+1)[(x^6+2)(x^7+3)]' \\ &= 5x^4(x^6+2)(x^7+3) + (x^5+1)[6x^5(x^7+3) + (x^6+2)7x^6] \end{aligned}$$

∎

(c) $f(x) = \dfrac{3x^2+2x+1}{x^3+2}$

Solution: $f(x) = \dfrac{(6x+2)(x^3+2) - (3x^2+2x+1)3x^2}{(x^3+2)^2}$ ∎

Question 2.

(a) Find an equation of the tangent line to $y = 3\sqrt{x} + 4$ at the point $(4, 10)$.

Solution: $y' = \dfrac{3}{2\sqrt{x}}$, so at $x = 4$, the slope of the tangent line is $\dfrac{3}{2\sqrt{4}} = \dfrac{3}{4}$. An equation of the tangent line at the point $(4, 10)$ is $y - 10 = \frac{3}{4}(x-4)$. ∎

(b) Find the point on the curve $y = x^3 + 1$ such that the tangent line to the curve at this point passes through the origin.

Solution: $y' = 3x^2$, so at any point $\langle c, c^3+1 \rangle$ on the curve, the slope of the tangent line is $3c^2$. If the tangent line through this point passes through the origin, its slope is also $\dfrac{c^3+1}{c}$. Setting these two expressions for the slope equal, we get:

$$\begin{aligned} 3c^2 &= \frac{c^3+1}{c} \\ 3c^3 &= c^3+1 \\ 2c^3 &= 1 \\ c &= \frac{1}{\sqrt[3]{2}} \end{aligned}$$

Therefore the tangent line at $\left\langle \frac{1}{\sqrt[3]{2}}, 3/2 \right\rangle$ passes through the origin. ∎

Question 3. Let $f(x) = (3x^2+2)\left(3+\frac{2}{x^2}\right)$.

Work out $f'(x)$ in two ways: (a) simplify the expression for $f(x)$ first, then differentiate; (b) differentiate first, then simplify. Compare your answers. (They must be the same!) Which route to a simplified answer was easier for you? Which method do you think would be best if $f(x) = x^{20}(3x^7 - 6x^6 + 7x^5 + 9x^4 - 3x^3 + 27)$; if $f(x) = (x^{20} - 7x^{15} + 9x^{12} - 36x^{10} + 12x^3 - 59)(3x^7 - 6x^6 + 7x^5 + 9x^4 - 3x^3 + 27)$?

Solution: This was more a question for thought than for computation. Anybody who actually worked out the derivatives of the two complicated expressions at the end wasn't paying attention.

(a) $(3x^2+2)\left(3+\frac{2}{x^2}\right) = 9x^2 + 6 + 6 + \frac{4}{x^2} = 9x^2 + \frac{4}{x^2} + 12$.

$f'(x) = 18x - \frac{8}{x^3}$.

(b) $f'(x) = 6x\left(3+\frac{2}{x^2}\right) + (3x^2+2)\left(\frac{-4}{x^3}\right)$.

$$6x\left(3+\frac{2}{x^2}\right) + (3x^2+2)\left(\frac{-4}{x^3}\right) = 18x + \frac{12}{x} - \frac{12}{x} - \frac{8}{x^3} = 18x - \frac{8}{x^3}.$$

A few comments from T. Kucera: personally, I prefer to simplify the expression first, then differentiate (at least when I know that I am going to have to do further work with the derivative!). The product rule, quotient rule, and chain rule all tend to produce more complicated expressions. Of the two "nasty" expressions in the question, the first one is not bad at all, and we should multiply that first factor of x^{20} through to give a simple (but long!) polynomial before doing anything else. The second expression is just plain awful no matter how we approach it. ∎

Question 4. Find $\frac{dy}{dt}$ if:

(a) $y = \sqrt{t + \sqrt{t + \sqrt{t}}}$

Solution: $$y' = \frac{1}{2\sqrt{t+\sqrt{t+\sqrt{t}}}}\left(1 + \frac{1}{2\sqrt{t+\sqrt{t}}}\left(1+\frac{1}{2\sqrt{t}}\right)\right)$$ ∎

(b) $y = \left(\frac{t}{t^4+1}\right)^5$

Solution: $$y' = 5\left(\frac{t}{t^4+1}\right)^4\left(\frac{1\cdot(t^4+1) - t\cdot(4t^3)}{(t^4+1)^2}\right)$$ ∎

(c) $y = t^3\sqrt[3]{1-t^3}$

Solution: $$y' = 3t^2\sqrt[3]{1-t^3} + t^3\frac{1}{3}\left(1-t^3\right)^{-2/3}\left(-3t^2\right)$$ ∎