MATH 1510 Tutorial Worksheet 2

September 18, 2007

SOLUTIONS

Question 1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to-\infty} \frac{3x^2+2x+1}{4x^2+3}$

Solution: $$\lim_{x\to-\infty} \frac{3x^2+2x+1}{4x^2+3} = \lim_{x\to-\infty} \frac{3+\frac{2}{x}+\frac{1}{x^2}}{4+\frac{3}{x^2}} = \frac{3}{4}$$ ∎

(b) $\displaystyle\lim_{x\to\infty} \frac{\sin(x+1)}{x-1}$

Solution: The function $\sin(x-1)$ is *bounded*: $-1 \leq \sin(x-1) \leq 1$ for all x. Furthermore, for $x > 1$, $\dfrac{1}{x-1} > 0$. Therefore, for all $x > 1$,

$$\frac{-1}{x-1} < \frac{\sin(x+1)}{x-1} < \frac{1}{x-1}$$

Since $\displaystyle\lim_{x\to\infty} \frac{1}{x-1} = 0$, by the squeeze theorem, $\displaystyle\lim_{x\to\infty} \frac{\sin(x+1)}{x-1} = 0$. ∎

(c) $\displaystyle\lim_{x\to-\infty} \sqrt{x^2+2x} - \sqrt{x^2-2x}$

Solution:

$$\begin{aligned}
\lim_{x\to-\infty} \sqrt{x^2+2x} - \sqrt{x^2-2x} &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+2x} - \sqrt{x^2-2x})(\sqrt{x^2+2x} + \sqrt{x^2-2x})}{(\sqrt{x^2+2x} + \sqrt{x^2-2x})} \\
&= \lim_{x\to-\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2(1+\frac{2}{x})} + \sqrt{x^2(1-\frac{2}{x})}} \\
& \quad [\text{for } x<0,\ \sqrt{x^2} = -x] \\
&= \lim_{x\to-\infty} \frac{4x}{-x\left(\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{2}{x}}\right)} \\
&= \lim_{x\to-\infty} \frac{-4}{\sqrt{1}+\sqrt{1}} \\
&= -2
\end{aligned}$$

∎

(d) $\displaystyle\lim_{x\to 2} \ln\left(\frac{x^2-4}{x-2}\right)$

Solution: The logarithm function is continuous, so:

$$\lim_{x\to 2} \ln\left(\frac{x^2-4}{x-2}\right) = \ln\left(\lim_{x\to 2} \frac{x^2-4}{x-2}\right) = \ln\left(\lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}\right) = \ln\left(\lim_{x\to 2}(x+2)\right) = \ln 4$$

∎

Question 2. The graph of $y = \dfrac{3x^2 + 2\sin(x)}{x+1}$ has an oblique asymptote. Find it.
Solution: We find oblique asymptotes by long division:

$$\begin{aligned}\frac{3x^2+2\sin(x)}{x+1} &= \frac{3x^2}{x+1} + \frac{2\sin(x)}{x+1} \\ &= \frac{3(x^2-1)+3}{x+1} + \frac{2\sin(x)}{x+1} \\ &= \frac{3(x^2-1)}{x+1} + \frac{3}{x+1} + \frac{2\sin(x)}{x+1} \\ &= 3(x-1) + \frac{3}{x+1} + \frac{2\sin(x)}{x+1}\end{aligned}$$

The second and the third terms of this expression both approach ∞ as x goes to either $+\infty$ or $-\infty$ (the third term, by the Squeeze Theorem), so what remains, $y = 3(x-1)$, is an oblique asymptote to the graph. ∎

Question 3. Consider the function

$$f(x) = \frac{1 - x - 2x^2}{x^2 - 3x + 2}$$

Find the x and y intercepts.
Find any horizontal asymptotes.
Find any vertical asymptotes (find the left and the right hand limits at these points).
Use this information to give a rough sketch of the graph of $y = f(x)$.

Solution: (The sketch is attached as a separate file).

For the y intercept, we set $x = 0$: $f(0) = \dfrac{1 - 0 - 2\cdot 0^2}{0^2 - 3\cdot 0 + 2} = \dfrac{1}{2}$.

For the x intercepts, we solve $f(x) = 0$. Since $f(x)$ is defined by a rational expression, we set the numerator equal to 0: $0 = 1 - x - 2x^2 = (1-2x)(1+x)$. So the x intercepts are at $x = \frac{1}{2}$ and $x = -1$.

For the horizontal asymptote, we take the limit at infinity:

$$\lim_{x\to\infty} \frac{1 - x - 2x^2}{x^2 - 3x + 2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{1}{x} - 2}{1 - \frac{3}{x} + \frac{2}{x^2}} = -2$$

There is a horizontal asymptote at $y = -2$.

Since $f(x)$ is defined by a rational expression, we find vertical asymptotes at points where the denominator is 0, namely $x = 1$ and $x = 2$. We test the behaviour of the asymptotes by left and right hand limits. Since we know the function is unbounded at these points, it is only necessary to decide if it goes to $+\infty$ or $-\infty$ on each side. For this it is helpful to know where each of the numerator and denominator is positive or negative.

$$1 - x - 2^{x^2} = (1-2x)(1+x)$$

x	$x < -1$	-1	$-1 < x < \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} < x$
$(1-2x)(1+x)$	$-$	0	$+$	0	$-$

$$x^2 - 3x + 2 = (x-1)(x-2)$$

x	$x < 1$	1	$1 < x < 2$	2	$2 < x$
$(x-1)(x-2)$	$+$	0	$-$	0	$+$

Putting all this information together and looking at values for x close to the asymptotes to either side, we find that

$$\lim_{x\to 1^-} f(x) = -\infty\,, \quad \lim_{x\to 1^+} f(x) = +\infty\,, \quad \lim_{x\to 2^-} f(x) = +\infty\,, \quad \lim_{x\to 2^+} f(x) = -\infty$$