ANNOTATED SOLUTIONS

DRAFT VERSION November 13

WARNING:

We think we have vitually all of the errors out of this file now.
If something doesn't make sense, let us know!

[15] 1. Find $\frac{dy}{dx}$ if:

[5] (a) $y = \cos^3(2x) + \tan(x^2)$

> **Solution:** $\frac{dy}{dx} = 3\cos^2(2x)(-\sin(2x)) \cdot 2 + \sec^2(x^2) \cdot 2x$
> Comment: There is no reason to try to use trigonometric identities to manipulate $\cos^3(2x)$ before differenting.

[4] (b) $y = \mathbf{e}^{5x} \ln(x^2 + 1)$

> **Solution:** $\frac{dy}{dx} = \mathbf{e}^{5x} \cdot 5 \cdot \ln(x^2 + 1) + \mathbf{e}^{5x} \frac{1}{x^2 + 1} 2x$
> Comment: There is nothing to be gained (and a lot to be lost) by using logarithmic differentiation for this problem.

[6] (c) $y = (2 + \cos(x))^{\sin(x)}$

> **Solution:** (Logarithmic differentiation)
>
> $$\begin{aligned} y &= (2 + \cos(x))^{\sin(x)} \\ \ln(y) &= \sin(x)\ln(2 + \cos(x)) \\ \frac{y'}{y} &= \cos(x)\ln(2 + \cos(x)) + \sin(x)\frac{1}{(2 + \cos(x)} \cdot (-\sin(x)) \\ y' &= (2 + \cos(x))^{\sin(x)} \left[\cos(x)\ln(2 + \cos(x)) - \frac{\sin^2(x)}{(2 + \cos(x)}\right] \end{aligned}$$
>
> Comment: In almost all cases, in the last line of this solution you must write out y explicitly, rather than just leaving it as the letter "y". The one exception is the case where we are using logarithmic differentiation to differentiate a complicated rational expression with several factors in the numerator and in the denominator, where the expression becomes so long that it is difficult to write.

General comment: Simplification was not required. In most problems, especially test problems, there is little profit in trying to simplify the question first.

[8] 2. Find an equation of the tangent line to the curve defined by $x\mathbf{e}^y - y\ln(x) = 5$ at the point $\langle\, \mathbf{e}^{-2},\, 2\,\rangle$.

Solution: (Implicit differentiation)

$$
\begin{aligned}
x\mathbf{e}^y - y\ln(x) &= 5 && (1)\\
(1\cdot\mathbf{e}^y + x\mathbf{e}^y y') - \left(y'\ln(x) + y\cdot\frac{1}{x}\right) &= 0 && (2)\\
\left(\mathbf{e}^2 + \mathbf{e}^{-2}\mathbf{e}^2 y'\right) - \left(y'\ln(\mathbf{e}^{-2}) + 2\cdot\frac{1}{\mathbf{e}^{-2}}\right) &= 0 && (3)\\
\left(\mathbf{e}^2 + y'\right) - \left(-2y' + 2\mathbf{e}^2\right) &= 0 && (4)\\
3y' &= \mathbf{e}^2 && (5)\\
y' &= \frac{\mathbf{e}^2}{3} && (6)
\end{aligned}
$$

Therefore an equation of the tangent line to the curve at $\langle\, \mathbf{e}^{-2},\, 2\,\rangle$ is

$$y - 2 = \frac{\mathbf{e}^2}{3}(x - \mathbf{e}^{-2})\,.$$

Alternatively: $y = \dfrac{\mathbf{e}^2}{3}x + \dfrac{5}{3}$.

Comments: Line (2) was the source of many problems, as the pair of parentheses around the second group was often omitted, thus changing the sign of $y\cdot\dfrac{1}{x}$. After line (2) the best method is to immediately substitute the values $x = \mathbf{e}^{-2}$ and $y = 2$, as the question does not ask you to find an expression for dy/dx. In general, although simplification is not required, you must complete simple arithmetic if you are going to use the answer in a subsequent step. So we expected you to do simple computations from the laws of exponents and logarithms: $\mathbf{e}^{-2}\mathbf{e}^2 = 1$, $\ln(\mathbf{e}^{-2}) = -2$, and $1/\mathbf{e}^{-2} = \mathbf{e}^2$.

For those of you that tried to do it, $\dfrac{dy}{dx} = \dfrac{\frac{y}{x} - \mathbf{e}^y}{x\mathbf{e}^y - \ln(x)}$

[4] 3. Suppose that $f(t)$ is continuous and differentiable for $t \geq 0$, that $2 \leq f'(t) \leq 3$ for $t \geq 1$, and that $f(1) = 5$. Use the Mean Value Theorem to estimate a range of possible values for $f(3)$.

Solution: Since $f(t)$ is continuous and differentiable, the Mean Value Theorem applies to the interval $1 \leq t \leq 3$.

Hence there is c, $1 < c < 3$ such that $f(3) - f(1) = f'(c)(3-1)$.

That is, $f(3) = 2f'(c) + 5$.

Since $2 \leq f'(c) \leq 3$, $2\cdot 2 + 5 \leq f(3) \leq 2\cdot 3 + 5$, that is, $9 \leq f(3) \leq 11$.

Comments: If you don't actually mention the Mean Value Theorem in your solution, you haven't answered the question! Justify its use (line 1), and explain the restrictions on c (line 2).

[6] 4. Newton's Method may be used to find an approximate value for $\sqrt{2}$, by approximating the positive root of the polynomial function $f(x) = x^2 - 2$.

[2] (a) Find the iteration formula giving x_{n+1} in terms of x_n for **this** problem.

Solution: If $f(x) = x^2 - 2$ then $f'(x) = 2x$.

Therefore, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{{x_n}^2 - 2}{2x_n} = \dfrac{{x_n}^2 + 2}{2x_n}$.

It is best to *simplify* any answer that you are going to have to use in a subsequent part of the problem; in this case it makes the arithmetic in part (c) just a little easier.

[1] (b) Make a reasonable choice of an **integer** (whole number) x_1 for this question. (There are two equally good answers; either is acceptable.)

Solution: We see $1^2 = 1 < 2 < 4 = 2^2$. Either choice of $x_1 = 1$ or $x_1 = 2$ will do for x_1.

Not only that, both choices give the same answer for x_2 in the next part!

If you want to be really formal about it and impress your professors, here is the "best" answer: $f(1) = 1$ and $f(2) = 2$, and so since f is continuous, by the Intermediate Value Theorem, there is c, $1 < c < 2$, such that $f(c) = 0$. Since 1 and 2 are adjacent integers, either will do for x_1.

[2] (c) Calculate x_2 and x_3. Give your answers as rational numbers in simplest form.

Solution:

$x_1 = 1$: $x_2 = \dfrac{1^2 + 2}{2 \cdot 1} = \dfrac{3}{2}$

$x_1 = 2$: $x_2 = \dfrac{2^2 + 2}{2 \cdot 2} = \dfrac{6}{4} = \dfrac{3}{2}$

$$x_3 = \frac{\left(\frac{3}{2}\right)^2 + 2}{2\left(\frac{3}{2}\right)} = \frac{17}{12}$$

[1] (d) Calculate ${x_3}^2$. How close is this to 2?

Solution:

$${x_3}^2 = \left(\frac{17}{12}\right)^2 = \frac{289}{144}$$

$$\frac{289}{144} - 2 = \frac{1}{144}$$

"How close" suggests "look at the difference between the two". You must have made some effort to answer "How close" to get the mark here.

Comment: The arithmetic in parts (c) and (d) should not have caused you so much difficulty!

[12] 5. COMMENT: Many of you like to use “sign diagrams” to work out the sign of f' in part (a) and the sign of f'' in part (b)—and this is generally how we prefer to teach it in class, as well. But it is rather difficult to *typeset* a good sign diagram. That is the ONLY reason that the solutions are explained in words here!

[6] (a) Let $f(x) = 3x^4 - 16x^3 + 24x^2$. Determine the critical points of f, determine the intervals on which $f(x)$ is increasing or decreasing, and classify the critical points as relative maxima, relative minima, or neither.

Solution: $f'(x) = 3 \cdot 4x^3 - 16 \cdot 3x^2 + 24 \cdot 2x = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x-2)^2$.

Therefore the critical numbers of f are $x = 0$ and $x = 2$.

Since $(x-2)^2$ is always positive, $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$.

Therefore f is decreasing for $\infty < x \leq 0$ and f is increasing for $0 \leq x < \infty$. Using interval notation: f is decreasing on $(\infty, 0]$ and f is increasing on $[0, \infty)$. (Either form of the answer is acceptable.)

There is a relative minimum at $x = 0$, $y = 0$.

(The critical point at $x = 2$, $y = 16$ does not give a relative extreme point.)

[6] (b) let $f(x) = x^4 - \frac{1}{5}x^5$. Determine the intervals on which $f(x)$ is concave up or concave down, and identify any inflection points.

Solution: $f'(x) = 4x^3 - \frac{1}{5} \cdot 5x^4 = 4x^3 - x^4$.

Therefore $f''(x) = 4 \cdot 3x^2 - 4x^3 = 4x^2(3 - x)$.

So $x = 0$ and $x = 3$ are potential inflection points.

Since x^2 is always positive, $f''(x) > 0$ if $x < 3$ and $f''(x) < 0$ if $x > 3$.

Therefore $f(x)$ is concave up for $-\infty < x < 0$ and for $0 < x < 3$, and concave down for $3 < x < \infty$. Using interval notation, $f(x)$ is concave up on $(-\infty, 0]$ and on $[0, 3]$, and concave down on $[3, \infty)$. (Either form of the answer is acceptable.)

There is an inflection point at $x = 3$, $y = 81 - \frac{243}{5} = \frac{162}{5}$.

There is **not** an inflection point at $x = 0$, $y = 0$.

COMMENT: In interval notation use brackets (“[”, “]”) when you want to *include* the endpoint of the interval, and parentheses (“(”, “)”) when you want to exclude the endpoint of the interval—and be sure you know which choice is appropriate in each case!

[15] 6. Let $f(x) = \dfrac{x^2}{4-x^2}$, (so $f'(x) = \dfrac{8x}{(4-x^2)^2}$ and $f''(x) = \dfrac{8(3x^2+4)}{(4-x^2)^3}$).

Answer the following questions by filling in the blanks, then give a neat sketch of the graph of $f(x)$. Note that "none" is a possible answer to any part.

ONE point for each part answered correctly, FIVE points for the sketch.

(a) Find the x-intercepts and the y-intercepts.

Solution: The only intercept is the origin: $x = 0$, $y = 0$.

(b) What symmetry does f have?

Solution: $f(-x) = \dfrac{(-x)^2}{4-(-x)^2} = \dfrac{x^2}{4-x^2} = f(x)$, so f is an even (symmetric) function.

(c) What are the horizontal asymptotes to the graph of $y = f(x)$?

Solution: $\lim_{x\to\pm\infty} f(x) = -1$, so $y = -1$ is the horizontal asymptote to the graph.

(d) What are the vertical asymptotes to the graph of $y = f(x)$?

Solution: $f(x)$ is not defined at $x = 2$ or at $x = -2$. The vertical asymptotes are $x = 2$ and $x = -2$.
Note that $x^2 \geq 0$ for all x; and that $4 - x^2$ is negative only when $-2 < x < 2$. So these are both odd vertical asymptotes, approaching $-\infty$ from inside $-2 < x < 2$ and approaching $+\infty$ from outside $-2 < x < 2$.

(e) On which intervals is f increasing?

Solution: The denominator of f' is always greater than or equal to zero, so f' is positive for $x > 0$ and f' is negative for $x < 0$.
Therefore, f is increasing on $0 \leq x < 2$ and $2 < x < \infty$, that is on $[0,2)$ and on $(2,\infty)$.
NOTE THAT 2 is *not* in the domain of the function and *cannot* be included in either of these intervals!

(f) On which intervals is f decreasing?

Solution: $-\infty < x < -2$ and $-2 < x \leq 0$, that is on $(-\infty,-2)$ and on $(-2,0]$.
NOTE THAT -2 is *not* in the domain of the function and *cannot* be included in either of these intervals!

(g) Find and classify the relative extreme points of f.

Solution: $f(x)$ has a local minimum at $\langle 0,\, 0 \rangle$.

(h) On which intervals is f concave up?

> **Solution:** The numerator of $f''(x)$ is always positive and the denominator is positive only when $-2 < x < 2$, so f'' is negative for $-\infty < x < -2$ and for $2 < x < \infty$; and f'' is positive for $-2 < x < 2$.
>
> $f(x)$ is concave up on $-2 < x < 2$, that is on $(-2, 2)$.
>
> NOTE THAT -2 and 2 are *not* in the domain of the function and *cannot* be included in this interval!

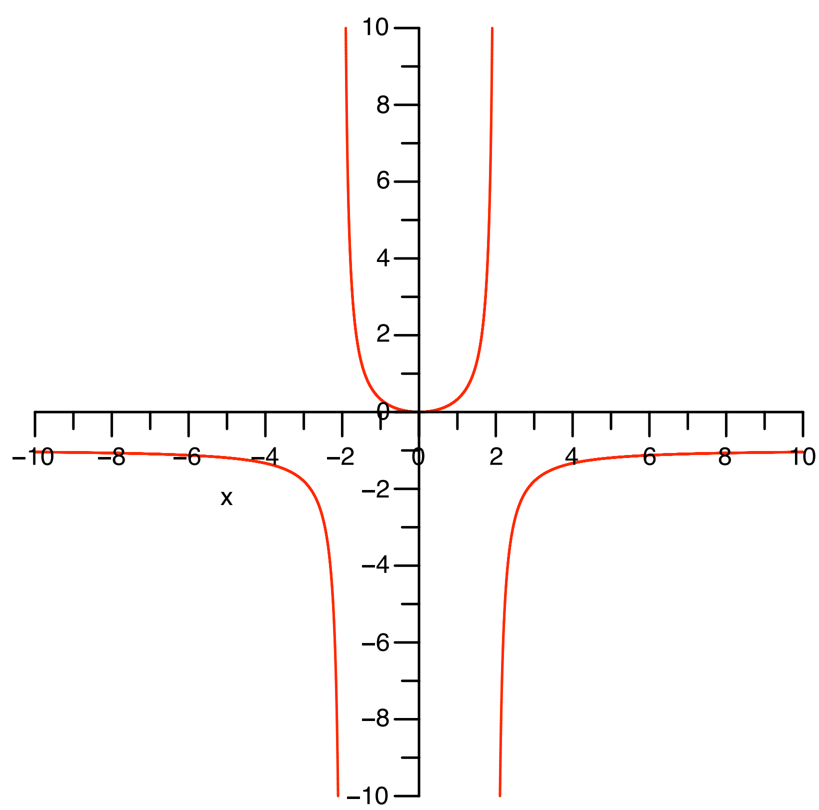
(i) On which intervals is f concave down?

> **Solution:** $f(x)$ is concave down on $-\infty < x < -2$ and on $2 < x < \infty$, that is on $(-\infty, -2)$ and on $(2, \infty)$.
>
> NOTE THAT -2 and 2 are *not* in the domain of the function and *cannot* be included in either of these intervals!

(j) What are the inflection points of f?

> **Solution:** $f''(x)$ is never equal to 0 ,and there are no points on the curve where $f''(x)$ is not defined, so f has no inflection points.

(k) Draw your sketch here.



Comment: It was not easy to include horizontal and vertical lines to show the asymptotes. We may update this sketch to show them, later.