MATH 1510 Test 1, Oct. 11, 2007
SOLUTIONS

[15] **Question 1.** Find each limit, if it exists.

Remarks: Many errors resulted from failing to include parentheses for grouping; this often results in errors in the multiplication of algebraic expressions.

"$\lim\limits_{x\to a}$" is an operator on functions or expressions; it has no meaning by itself. In particular, an expression like "$\lim\limits_{x\to 3} = -81$" does not mean anything mathematically and is *not* the answer to part (a)

(a)

$$\begin{aligned}
\lim_{x\to 3}\frac{3x^3-81}{3-x} &= \lim_{x\to 3}\frac{3\left(x^3-27\right)}{3-x}\\
&= \lim_{x\to 3}\frac{3(x-3)\left(x^2+3x+9\right)}{-(x-3)}\\
&= \lim_{x\to 3}(-3)\left(x^2+3x+9\right)\\
&= -3\cdot 27 \;=\; -81
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to -2^-}\frac{|\,x+2\,|}{x^3-4x} &= \lim_{x\to -2^-}\frac{-(x+2)}{x(x-2)(x+2)}\\
&= \lim_{x\to -2^-}\frac{-1}{x(x-2)}\\
&= \frac{-1}{(-2)(-4)} = \frac{-1}{8}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x}\\
&= \lim_{x\to\infty}\frac{\left(x^2+4x\right)-x^2}{\sqrt{x^2+4x}+x}\\
&= \lim_{x\to\infty}\frac{4x}{\sqrt{x^2(1+4/x)}+x}\\
&= \lim_{x\to\infty}\frac{4x}{x\sqrt{1+4/x}+x}\\
&= \lim_{x\to\infty}\frac{4}{\sqrt{1+4/x}+1} \;=\; \frac{4}{1+1} \;=\; 2
\end{aligned}$$

[20] **Question 2.** Find $f'(x)$ if: [DO NOT SIMPLIFY YOUR ANSWERS]
Remark: Generally speaking, we ignore attempts at simplification, but if you do attempt to simplify and you radically changed the nature of the answer as a result of algebra errors, we have to take note of that fact and deduct marks.

Common errors in parts (c) and (d) resulted from the misuse—or failure to use—parentheses properly.

(a) $f(x) = 3x^4 + \sqrt[5]{x^2} + \dfrac{1}{x} + \pi^3 = 3x^4 + x^{2/5} + x^{-1} + \pi^3$.

$$f'(x) = 4 \cdot 3x^3 + \frac{2}{5}x^{-3/5} - 1 \cdot x^{-2}$$

(b) $f(x) = x^3\sqrt{x^2+4}$

$$f'(x) = 3x^2\sqrt{x^2+4} + x^3\left(\frac{1}{2}\right)\left(x^2+4\right)^{-1/2}(2x)$$

(c) $f(x) = \dfrac{x^5+3x^2+2}{4-2x^3}$

$$f'(x) = \frac{\left(5x^4 + 2 \cdot 3x\right)\left(4-2x^3\right) \; - \; \left(x^5+3x^2+2\right)\left(-3 \cdot 2x^2\right)}{\left(4-2x^3\right)^2}$$

(d) $f(x) = \left[x^4 - (x+5)^{50}\right]^{10}$

$$f'(x) = 10\left[x^4 - (x+5)^{50}\right]^9 \cdot \left[4x^3 - 50(x+5)^{49}\right]$$

[5] **Question 3.** Find the value of $f'''(2)$ if $f(x) = x^4 - 3x^3 + 2x - 100$.
Remark: *First* you have to find $f'''(x)$, *then* calculate the value at $x = 2$.

$$\begin{aligned} f(x) &= x^4 - 3x^3 + 2x - 100 \\ f'(x) &= 4x^3 - 9x^2 + 2 \\ f''(x) &= 12x^2 - 18x \\ f'''(x) &= 24x - 18 \\ f'''(2) &= 48 - 18 \;=\; 30 \end{aligned}$$

[7] **Question 4.** Use the *definition of the derivative* to find $f'(x)$ if $f(x) = \dfrac{x}{x+1}$.

REMARK: Since this was a *formal* question, marks *were* deducted for formal faults. In particular, we noted (with some distress) that many of you chose to omit equal signs and/or limit symbols from your calculations. In addition, note that "$\lim\limits_{x\to a}$" is an operation applied to a function or to an algebraic expression; it cannot occur in isolation, nor can it be followed by an equal sign.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{(x+h)+1} - \dfrac{x}{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h)(x+1)-(x+h+1)x}{(x+h+1)(x+1)}}{h} \\
&= \lim_{h\to 0} \frac{(x^2+hx+x+h)-(x^2+hx+x)}{h(x+h+1)(x+1)} \\
&= \lim_{h\to 0} \frac{h}{h(x+h+1)(x+1)} \\
&= \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} \\
&= \frac{1}{(x+1)^2}
\end{aligned}$$

[7] **Question 5.** Use *limits* to determine the value(s) of k if the function f defined by

$$f(x) = \begin{cases} 2x^2 + 5k & x \geq 2 \\ kx + 23 & x < 2 \end{cases}$$

is continuous at $x = 2$.

REMARK: Since this was a *formal* question, marks *were* deducted for formal faults. In particular, we noted (with some distress) that many of you chose to omit equal signs and/or limit symbols from your calculations. In addition, note that "$\lim\limits_{x \to a}$" is an operation applied to a functions or to an algebraic expression; it cannot occur in isolation, nor can it be followed by an equal sign.

In addition, on this question in particular, we noted that the work was often presented in a very disorganized manner, with bits and pieces of the solution spread around the work area and with no connections shown between the different parts of the solution.

Solution: First note the *definition* of continuity: $f(x)$ is *continuous* at $x = 2$ if $\lim\limits_{x \to 2} f(x) = f(2)$.

From the question, $f(2) = 2 \cdot 2^2 + 5k = 8 + 5k$.

Since $f(x)$ is defined by cases, with the cases separating at $x = 2$, we must evaluate $\lim\limits_{x \to 2} f(x)$ by left-hand and right-hand limits:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} kx + 23 = 2k + 23$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x^2 + 5k = 8 + 5k$$

For $\lim\limits_{x \to 2} f(x)$ to exist, the left-hand and right-hand limits must be equal: $8 + 5k = 2k + 23$, so $3k = 15$ and therefore $k = 5$. When $k = 5$, $\lim\limits_{x \to 2} f(x) = 33$ and $f(2) = 8 + 5 \cdot 5 = 33$, so $f(x)$ is continuous at $x = 2$.

[6] **Question 6.**

(a) Find an equation of the tangent line to the curve $y = 4x - x^2$ at the point on the curve with x-coordinate 3.

Solution: When $x = 3$, $y = 4 \cdot 3 - 3^2 = 3$.

$y' = 4 - 2x$, so when $x = 2$, $y' = 4 - 2 \cdot 3 = -2$: the slope of the tangent line at $\langle 3, 3 \rangle$ is -2.

An equation of the tangent line is $y - 3 = -2(x - 3)$.

[Other forms of this equation are $y = -2x + 9$ and $2x + y = 9$.]

(b) Find the area of the triangle formed by the tangent line, in part (a) above, and the coordinate axes [i.e. the x-axis and the y-axis].

Solution: The intercepts of the tangent line (with the coordinate axes) are easily found: when $x = 0$, $y = 9$; and when $y = 0$, $x = 9/2$. So the triangle is a right triangle wiht base $9/2$ and height 9, therefore the area of the triangle is $\left(\frac{1}{2}\right)\left(\frac{9}{2}\right) 9 = \frac{81}{4}$.