WARNING Although we have made reasonable efforts to proof-read these solutions, we do not guarantee that this solution set is free of error: given past experience, we would be surprised if there were not a few significant typoes here and there!

[9] 1. Evaluate the following limits:

[4] (a) $\lim_{x \to 2} \frac{x^2 - 4}{8 - x^3}$

Solution:

$$\begin{aligned}\lim_{x \to 2} \frac{x^2 - 4}{8 - x^3} &= \lim_{x \to 2} \frac{(x-2)(x+2)}{(2-x)(4+2x+x^2)} \\ &= \lim_{x \to 2} -\frac{x+2}{4+2x+x^2} \\ &= -\frac{4}{12} \quad = \quad -\frac{1}{3}\end{aligned}$$

[5] (b) $\lim_{x \to -\infty} \frac{\sqrt{3x^2+4}}{2x+5}$

Solution:

$$\begin{aligned}\lim_{x \to -\infty} \frac{\sqrt{3x^2+4}}{2x+5} &= \lim_{x \to -\infty} \frac{|\,x\,| \sqrt{3+4/x^2}}{x(2+5/x)} \qquad (\sqrt{x^2}) = |\,x\,|) \\ &= \lim_{x \to -\infty} \frac{-x\sqrt{3+4/x^2}}{x(2+5/x)} \qquad (x \to -\infty) \\ &= \lim_{x \to -\infty} \frac{-\sqrt{3+4/x^2}}{(2+5/x)} = -\frac{\sqrt{3}}{2}\end{aligned}$$

[15] 2. Find $\frac{dy}{dx}$ in each case (DO NOT SIMPLIFY YOUR ANSWERS):

[4] (a) $y = \frac{\sec(x)}{x^4 + 10}$

Solution: $\frac{dy}{dx} = \frac{\sec(x)\tan(x)(x^4+10) \; - \; (4x^3)\sec(x)}{(x^4+10)^2}$

[4] (b) $y = \mathbf{e}^{-x} \cos\left(\frac{\pi}{4}x\right)$

Solution: $\frac{dy}{dx} = -\mathbf{e}^{-x}\cos\left(\frac{\pi}{4}x\right) + \mathbf{e}^{-x}\left(-\sin\left(\frac{\pi}{4}x\right)\right)\left(\frac{\pi}{4}\right)$

[3] (c) $y = (x^3 + 3)^{10}$

Solution: $\frac{dy}{dx} = 10(x^3+3)^9(3x^2)$

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[4] (d) $y = \ln(3^x + x^2)$

Solution: $\dfrac{dy}{dx} = \dfrac{1}{3^x + x^2} \cdot (3^x \ln(3) + 2x)$

[19] 3. Evaluate the following indefinite and definite integrals:

[3] (a) $\int (5x - 14)^{10}\, dx$

Solution: $\int (5x-14)^{10}\, dx = \dfrac{1}{5} \cdot \dfrac{1}{11}(5x-14)^{10} + C$

[6] (b) $\in 0^1 \dfrac{x}{(x^2+4)^2}\, dx$

Solution:

$$\int_0^1 \frac{x}{(x^2+4)^2}\, dx \qquad \begin{pmatrix} u = x^2+4 & du = 2x\, dx \\ x = 0,\ u = 4; & x = 1,\ u = 5 \end{pmatrix}$$

$$= \frac{1}{2}\int_{u=4}^{u=5} \frac{du}{u^2}$$

$$= \frac{1}{2}\left(\frac{u^{-1}}{-1}\right)\Bigg|_4^5$$

$$= -\frac{1}{2}\left(\frac{1}{5} - \frac{1}{4}\right) \quad = \quad \frac{1}{40}$$

[4] (c) $\int \dfrac{1}{x \ln(x)}\, dx$

Solution:

$$\int \frac{1}{x\ln(x)}\, dx \quad = \quad \left(u = \ln(x),\ du = \frac{dx}{x}\right)$$

$$= \int \frac{du}{u}$$

$$= \ln|\,u\,| + C$$

$$= \ln|\ln(x)\,| + C$$

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[6] (d) $\int_0^5 x\sqrt{9-x}\,dx$

Solution:

$$\int_0^5 x\sqrt{9-x}\,dx \qquad \begin{pmatrix} u = 9-x \ (x = 9-u) & du = -dx \\ x=0,\ u=9; & x=5,\ u=4; \end{pmatrix}$$

$$= \int_9^4 (9-u)u^{1/2}(-du)$$

$$= \int_4^9 \left(9u^{1/2} - u^{3/2}\right)\,du$$

$$= 9\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\Big|_4^9$$

$$= (6\cdot 27 - \frac{2}{5}\cdot 243) - (6\cdot 8 - \frac{2}{5}\cdot 32) \quad = \quad \frac{148}{5}$$

Solution:

$$\int_0^5 x\sqrt{9-x}\,dx \qquad \begin{pmatrix} u^2 = 9-x \ (x = 9-u^2) & 2u\,du = -dx \\ x=0,\ u=3; & x=5,\ u=2; \end{pmatrix}$$

$$= \int_3^2 (9-u^2)u(-2u\,du)$$

$$= 2\int_2^3 \left(9u^2 - u^4\right)\,du$$

$$= \left(9\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_2^3$$

$$= 2(3\cdot 81 - \frac{243}{5}) - (3\cdot 8 - \frac{32}{5}) \quad = \quad \frac{148}{5}$$

[14] 4. A particle moves on the x-axis with acceleration $a(t) = (4-6t)\text{m/s}^2$. At time $t = 0\text{s}$ the position is $x = 3\text{m}$ and the velocity is 4m/s.

[4] (a) What is the velocity of the particle at $t = 1\text{s}$?

Solution: $v(t) = \int a(t)\,dt = 4t - 6\frac{t^2}{2} + C$

$v(0) = 4 = C$

$v(t) = 4 + 4t - 3t^2$

$v(1) = 4 + 4 - 3 = 5$

The particle has velocity 5m/s at time $t = 1\text{s}$.

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[4] (b) What is the position of the particle at $t = 1$s?

Solution: $x(t) = \int v(t)\,dt = 4t + 4\frac{t^2}{2} - 3\frac{t^3}{3} + D$
$x(0) = 3 = D$
$x(t) = 3 + 4t + 2t^2 - t^3$
$x(1) = 3 + 4 + 2 - 1 = 8$
The particle is at position 8m on the x-axis at time $t = 1$s.

[3] (c) Is the particle speeding up or slowing down at $t = 1$s? (Explain!)

Solution: $v(1) = 5$m/s, $a(1) = 4 - 6 = -2$m/s^2.
Since the velocity is positive and the acceleration is negative, the particle is slowing down at time $t = 1$s.

[3] (d) Is the particle speeding up or slowing down at $t = 3$s? (Explain!)

Solution: $v(3) = (4 + 12 - 27)$m/s $= 11$m/s, $a(3) = 4 - 18 = -14$m/s^2.
Since the velocity is negative and the acceleration is negative, the particle is speeding up at time $t = 3$s.

[8] 5. Find the absolute maximum value and the absolute minimum value of $g(t) = t^2\mathbf{e}^{-t}$ on the interval $[-1, 3]$.

Solution: $g'(t) = 2t\mathbf{e}^{-t} + t^2(-\mathbf{e}^{-t}) = (2-t)t\mathbf{e}^{-t}$.
The critical points occur at $t = 0$, $t = 2$, both in the given interval.

t	-1	0	2	3
$g(t)$	$\mathbf{e}$	0	$4\mathbf{e}^{-2}$	$9\mathbf{e}^{-3}$

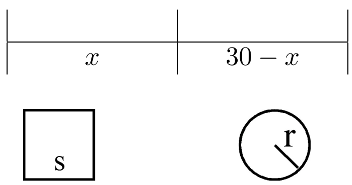
The minimum value of g is clearly 0 [at $t = 0$].
We have to decide which of the remaining values is the maximum.
Since $2 < \mathbf{e}$, $\mathbf{e}^{-2} < \frac{1}{4}$ and so $4\mathbf{e}^{-2} < 1 < \mathbf{e}$.
Since $2 < \mathbf{e}$, $\mathbf{e}^{-3} < \frac{1}{8}$ and so $9\mathbf{e}^{-2} < \frac{9}{8} < \mathbf{e}$.
Therefore the maximum value of g is $\mathbf{e}$ [at $t = -1$].

[8] 6. Consider the following word problem:

> "A jeweler is going to cut a piece of gold wire 30cm long into two pieces. One piece will be bent into a square, the other piece will be bent into a circle. Find the length of the piece that will be bent into a square so that the total area enclosed by the square and the circle is maximized."

DO NOT SOLVE THIS WORD PROBLEM! Just set up the equivalent mathematical question: draw a neat sketch illustrating the situation described; identify the variables involved; set up the equations described by the problem; and find a function of one variable to be maximized. State any restrictions on the variables involved (that is, determine the domain of the function).

Solution:



Let x be the length of the piece used to make the square, so $30 - x$ is the length of the piece used to make the circle.
The length of the side of the square is $s = \dfrac{x}{4}$.
The radius of the circle is $r = \dfrac{30-x}{2\pi}$.

Let A be the total area enclosed:

$$A(x) = s^2 + \pi r^2 = \frac{x^2}{16} + \pi\left(\frac{30-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(30-x)^2}{4\pi}$$

Physical constraint: $0 \le x \le 30$.

[10] 7. A particle moves on the parabola $y = x^2 - 2x + 2$, x and y measured in metres. When the particle is at the point $(2, 2)$, its x-coordinate is decreasing at $\frac{1}{20}$m/s. How fast is the distance from the particle to the point $(3, 0)$ changing at this time?

Solution: Let s be the distance (metres) from the particle to the point $(2, 2)$.
Then $s^2 = (x-3)^2 + (y-0)^2 = (x-3)^2 + (x^2-2x+2)^2$.
[Comment: there is really no point in simplifying this further, although we could!]

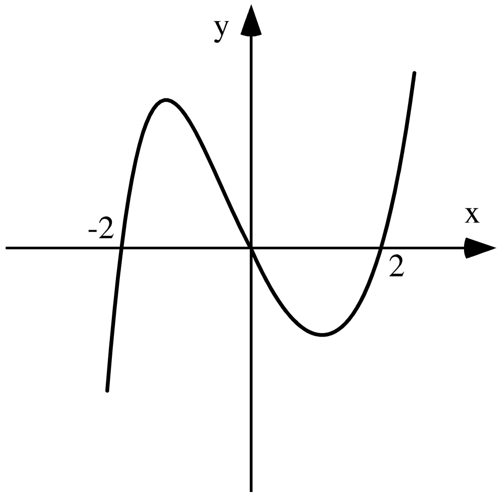
Therefore, $2s\dfrac{ds}{dt} = 2(x-3)\dfrac{dx}{dt} + 2(x^2-2x+2)(2x-2)\dfrac{dx}{dt}$.

When $x = 2$ and $y = 2$, $s^2 = (-1)^2 + 2^2 = 5$, so $s = \sqrt{5}$. $\dfrac{dx}{dt} = -\dfrac{1}{20}$.

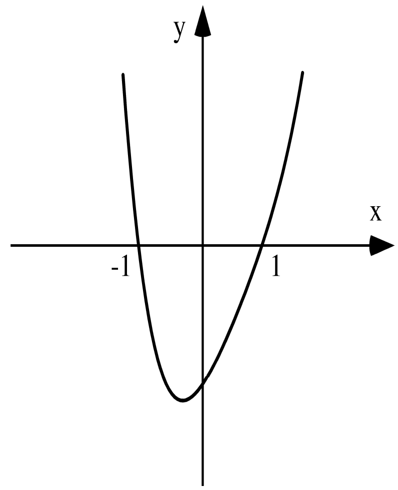
Therefore $\sqrt{5}\dfrac{ds}{dt} = (-1)(-\dfrac{1}{20}) + (2)(2)(-\dfrac{1}{20}) = \dfrac{-3}{20\sqrt{5}}$.

The distance from the particle to the point $(2, 2)$ is decreasing at $\dfrac{3}{20\sqrt{5}}$m/s.

[17] 8. Consider the following two sketches and table of information about the function $f(x)$, which is defined and continuous on $(-\infty, \infty)$:



$y = f'(x)$



$y = f''(x)$

x	-3	-2	-1	0	1	2	3
$f(x)$	0	-3	-2	0	$-1/2$	-1	0

This information includes EVERYTHING that is "interesting" about the curve. Please note that there are no "tricks" hidden in minor flaws in the sketches!

(a) On what intervals is f increasing? $[-2, 0]$, $[2, \infty)$

(b) On what intervals is f decreasing? $(-\infty, -2]$, $[0, 2]$

(c) Find the coordinates of all the local maxima of f. $(0, 0)$

(d) Find the coordinates of all the local minima of f. $(-2, -3)$, $(2, -1)$

(e) On what intervals is f concave up? $(-\infty, -1]$, $[1, \infty)$

(f) On what intervals is f concave down? $[-1, 1]$

(g) Find the coordinates of all the inflection points of f. $(-1, -2)$, $(1, \frac{-1}{2})$

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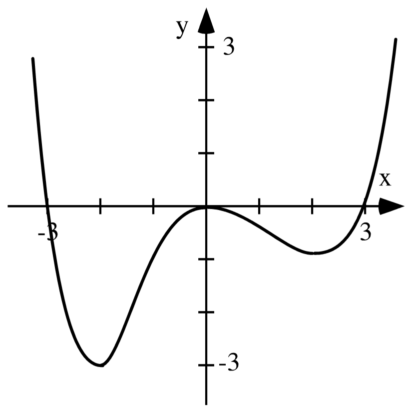
[5] (h) Give a rough sketch of the graph of $y = f(x)$.



- axes, labels, presentation
- critical points
- inflection points
- increasing/decreasing
- concavity