MATH 1510 Problem Set 3

November 6, 2007

SOLUTIONS

Question 1. Find the absolute maximum and absolute minimum (if any) of each function on the interval indicated:

(a) $f(x) = x^4 - 4x^2 + 2$, on the interval $[-1, 3]$,

Solution: $f'(x) = 4x^3 - 4 \cdot 2x = 4x(x-2)$. The critical points are at $x = 0$ and at $x = 2$, both in the interval. $f(-1) = -1$, $f(0) = 2$, $f(2) = 2$, and $f(3) = 47$.

The absolute maximum value of $f(x)$ on the interval $[-1, 3]$ is 47 and the absolute minimum value is -1.

(b) $g(x) = x - 2\cos(x)$, on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

Solution: $g(x) = 1+2\sin(x)$. The critical points of g are at $x = -\frac{\pi}{6} + 2n\pi$ and at $x = -\frac{5\pi}{6} + 2n\pi$ for n any integer; in our interval the only critical point is at $x = -\frac{\pi}{6}$. $g(-\frac{\pi}{2}) = -\frac{\pi}{2}$, $g(-\frac{\pi}{6}) = -\frac{\pi}{6} - \sqrt{3}$, and $g(\frac{\pi}{2}) = \frac{\pi}{2}$.

Since $\sqrt{3} > \frac{\pi}{2}$, the middle value is the smallest and so the absolute maximum value of $g(x)$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is $\frac{\pi}{2}$ and the absolute minimum value is $-\frac{\pi}{6} - \sqrt{3}$.

(c) $h(x) = (x-1)^2\mathbf{e}^{-x}$ on the interval $[0, \infty)$.

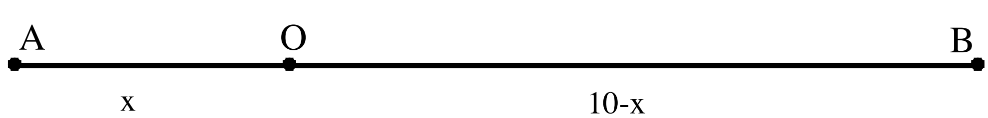
(Remember that if $p(x)$ is any polynomial then $\lim\limits_{x\to\infty} p(x)\mathbf{e}^{-x} = 0$.)

Solution: $h'(x) = 2(x-1)\mathbf{e}^{-x} + (x-1)^2\left(-\mathbf{e}^{-x}\right) = (x-1)(2-(x-1))\mathbf{e}^{-x} = (x-1)(3-x)\mathbf{e}^{-x}$. The critical points are at $x = 1$ and $x = 3$, both within the interval we are considering. $h(0) = 1$, $h(1) = 0$, and $h(3) = 4\mathbf{e}^{-3}$. Certainly $h(3) < 1$. Since $h(x) \geq 0$ everywhere and $\lim_{x\to\infty} h(x) = 0$, the absolute maximum value of $h(x)$ on the interval $[0, \infty)$ is 1 and the absolute minimum value is 0.

Question 2. Set up the equation(s) required to solve the following word problems. Draw any diagrams necessary, identifying variables, and clearly state the function to be maximized or minimized, with any constraints on the variables. DO NOT SOLVE THE PROBLEM!

(a) The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 m apart, where should an object be placed on the line between the two sources so as to receive the minimum possible illumination?

Solution:



The first sentence tells us that if I represents illunination, s represents the strength of the source, and x represents the distance from the source, then for some constant k, $I = k\dfrac{s}{x^2}$.

So for this problem, let (as in the diagram) **A** be the stronger light source, **B** be the weaker source, and x be the distance of the object from the stronger source (so that $10 - x$ is the distance of the object from the weaker source). Therefore $0 \leq x \leq 10$.

If s is the strength of light source **B**, then $3s$ is the strength of light source **A**.

Let y be the total illumination on the object.

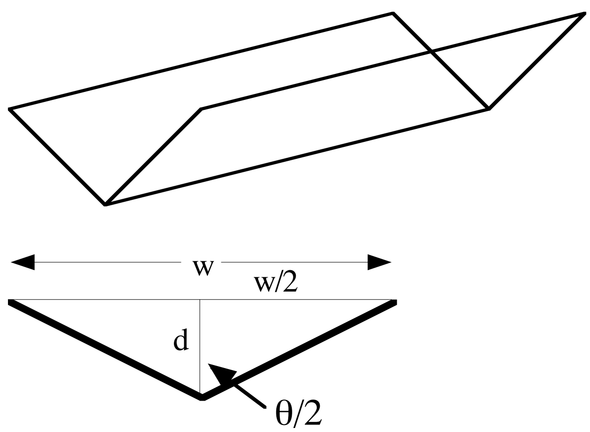
Then $y = k\dfrac{3s}{x^2} + k\dfrac{3}{(10-x)^2}$.

The problem then asks "Find x , $0 \leq x \leq 10$, so that $y = k\dfrac{3s}{x^2} + k\dfrac{3}{(10-x)^2}$. is a minimum."

COMMENT: We neglected to include appropriate units of measurement for illumination in this question; it doesn't affect the solution but was sloppy on our part.

(b) A long piece of sheet-metal 0.5 m wide is to be bent in half lengthwise to form a trough whose cross-section is an isosceles triangle. What is the angle at the bottom of the trough that gives the maximum possible cross-sectional area, and therefore the greatest volume for the trough?

Solution:



Let θ be the angle at the bottom of the trough; therefore since the metal is bent to form an isosceles triangle, the angle between the side of the trough and the vertical is $\theta/2$. Clearly $0 \leq \theta \leq \pi$.

Let d be the depth of the trough, and let w be the width of the trough, in metres. Let A be the cross-sectional area of the trough, in m^2 .

Then $A = \dfrac{1}{2}wd$.

Since the metal is bent to form an isosceles triangle, the two sides of the trough are of equal length: 0.25 m . Therefore (by considering the triangle on the right of the diagram) we find that $d = 0.25\cos(\theta/2)$ and that $w/2 = 0.25\sin(\theta/2)$.

Therefore $A = \dfrac{1}{2}(2 \cdot 0.25\sin(\theta/2)) \cdot (0.25\cos(\theta/2)) = 0.0625\sin(\theta/2)\cos(\theta/2) = 0.3125\sin(\theta)$. (The latter step follows by the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, or equivalently, $\dfrac{1}{2}\sin(\theta) = \sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right)$.

The problem then asks, "Find θ , $0 \leq \theta \leq \pi$, so that $A = 0.3125\sin(\theta)$ is a maximum."

COMMENT: Note that after all this work, the problem in question 2(b) is completely trivial by trigonometry: the answer is $\theta = \pi/2$.

Question 3.

(a) An object moves along the x-axis so that its position at time t, $0 \leq t \leq 1$ is given by $x(t) = 9t^4 - 16t^3 + 8t^2$, where position is given in metres and time is given in seconds. Find the following:

1. The time intervals on which the velocity is increasing and on which the velocity is decreasing.

   Solution: The velocity is give by $v(t) = x'(t) = 36t^3 - 48t^2 + 16t = 4t(9t^2 - 12t + 4) = 4t(3t - 2)^2$. The acceleration is given by $a(t) = v'(t) = x''(t) = 108t^2 - 96t + 16 = 4(27t^2 - 24t + 4) = 4(9t - 2)(3t - 2)$. The critical points of the *velocity* are at $x = 2/9$ and $x = 2/3$. Since the graph of $x''(t)$ is a parabola opening upwards, x'' is positive for $x < 2/9$ and for $x > 2/3$; x'' is negative for $2/9 < x < 2/3$. Therefore the velocity is **increasing** on the time intervals $t \leq 2/9$ and $2/3 \leq t$, and **decreasing** on the time interval $2/9 \leq t \leq 2/3$.

2. The time intervals on which the speed is increasing and on which the speed is decreasing.

   Solution: The speed is the absolute value of the velocity, $s(t) = | v(t) |$, and the rate of change of the speed is given by

   $$\frac{d}{dt}s(t) = \frac{d}{dt} | v(t) | = \operatorname{sgn}(v(t))v'(t) = \operatorname{sgn}(v(t))a(t),$$

   where sgn is the signum function. (We use the known formula for the derivative of absolute value, and the chain rule.)

   Now $v(t) = 0$ if $t = 0$ or if $t = 2/3$; and $v(t)$ changes from negative to positive as t increases through 0. Therefore the sign of the derivative of the speed can be read off from the solution to the previous part: it is negative for $t < 0$ and for $2/9 < t < 2/3$, and positive for $0 < t < 2/9$ and $2/3 < t$. Therefore the speed is increasing on the time intervals $[0, 2/9]$ and $[2/3, \infty]$; and the speed is decreasing on the time intervals $(-\infty, 0]$ and $[2/9, 2/3]$.

3. The maximum and the minimum value of the velocity on the time interval $-\frac{1}{3} \leq t \leq 1$.

   Solution: The extreme values occur at end points of the interval or critical numbers *of the velocity*. These are $t = -1/3,\ 2/9, /, 2/3, /, 1$. $v(-1/3) = -12$, $v(2/9) = 128/81$, $v(2/3) = 0$, $v(1) = 4$.

   Therefore the absolute maximum vale of the velocity on the time interval $-\frac{1}{3} \leq t \leq 1$ is 4 m/s and the absolute minimum value of the velocity is -12 m/s.

4. The maximum and the minimum value of the speed on the time interval $-\frac{1}{3} \leq t \leq 1$.

   Solution: The speed has an additional critical point, where the derivative of the speed is not defined, namely at $t = 0$. (The derivative is also not defined at $t = 2/3$ which was already under consideration.) The speed is the absolute value of the velocity, so the speed at $t = 0$ is the only remaining value to compute. But this is just 0 as well. Therefore the absolute maximum value of the speed is 12 m/s and the absolute minimum value is 0 m/s.

5. The average velocity over the time interval $-\frac{1}{3} \leq t \leq 1$.

   Solution: "Average velocity" just means total displacement divided by time elapsed (it is *not* the average of the values of the velocity at the endpoints of the interval):

   $$\frac{x(1) - x(-1/3)}{1 - (-1/3)} = \frac{1 - \frac{43}{27}}{4/3} = -\frac{4}{9} \text{ m/s}$$

(b) An object moves along the x-axis so that its position at time t, $t \geq 0$, is given by a function $x(t) = at^3 + bt^2 + ct + d$. It starts at the origin with velocity $2\,\mathrm{m/s}$ and acceleration $-6\,\mathrm{m/s^2}$, and returns to the origin at time $t = 2\,\mathrm{s}$.

1. Find the values of a, b, c, and d.

   Solution: The problem gives us information about the numerical value of the position, velocity and acceleration.

   $x(0) = d$, so $d = 0$.

   $v(t) = x'(t) = 3at^2 + 2bt + c$, and $v(0) = c$, so $c = 2$.

   $a(t) = v'(t) = 6at + 2b$, and $a(0) = 2b$, so $b = -3$.

   $0 = x(2) = 8a + 4b + 2c + d = 8a - 12 + 4 + 0 = 8a - 8$, so $a = 1$.

   Therefore $x(t) = t^3 - 3t^2 + 2t$.

2. Does the object pass through the origin at any other time?

   Solution: $x(t) = 0$ if $0 = t^3 - 3t^2 + 2t = t(t^2 - 3t + 2) = t(t-1)(t-2)$. Therefore the object also passes through the origin at $t = 1$ s.

3. What is the speed of the object at each time that it passes through the origin?

   Solution: Speed is the absolute value of the velocity, $|3t^2 - 6t + 2|$.

   At time $t = 0$, the speed is $2\,\mathrm{m/s}$.

   At time $t = 1$, the speed is $|3 - 6 + 2| = 1\,\mathrm{m/s}$.

   At time $t = 2$, the speed is $|12 - 12 + 2| = 2\,\mathrm{m/s}$.

4. What is the acceleration of the object at each time that it passes throught the origin?

   Solution: The acceleration is $a(t) = 6t - 6$.

   At time $t = 0$, the acceleration is $-6\,\mathrm{m/s^2}$.

   At time $t = 1$, the acceleration is $0\,\mathrm{m/s^2}$.

   At time $t = 2$, the acceleration is $6\,\mathrm{m/s^2}$.

5. On what interval (if any) is the object moving from right to left?

   Solution: The object is moving from right to left if the velocity is negative. $v(t) = 3t^2 - 6t + 2 = 0$ if $t = \dfrac{6 \pm \sqrt{36-24}}{6} = 1 \pm \dfrac{\sqrt{3}}{3}$. [Sorry about that: we let one "bad" pair of numbers creep in to the solutions at the very end!] Since the graph of $v(t)$ is a parabola opening upwards, $v(t)$ is negative between these two roots, and so the object is moving from right to left for $1 - \dfrac{\sqrt{3}}{3} \leq t \leq 1 + \dfrac{\sqrt{3}}{3}$.