MATH 1510 Problem Set 1

October 1, 2007

SOLUTIONS

Question 1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to -3/2} \frac{8x^3+27}{2x-3}$

Solution: Since the denominator does not approach 0, this limit can be evaluated by direct substitution:

$$\lim_{x\to -3/2} \frac{8x^3+27}{2x-3} = \frac{0}{-6} = 0$$

$\blacksquare$

(b) $\displaystyle\lim_{x\to -3/2} \frac{8x^3+27}{2x+3}$

Solution: Both the numerator and denominator approach 0 as x approaches $3/2$, so we use algebraic manipulation to solve the limit:

$$\lim_{x\to -3/2} \frac{8x^3+27}{2x+3} = \lim_{x\to -3/2} \frac{(2x+3)(4x^2+6x+9)}{2x+3} = \lim_{x\to -3/2}(4x^2+6x+9) = 9$$

$\blacksquare$

(c) $\displaystyle\lim_{x\to -3^-} \frac{x^2+x-6}{x+3}$

Solution: Both the numerator and denominator approach 0 as x approaches -3, so we use algebraic manipulation:

$$\lim_{x\to -3^-} \frac{x^2+x-6}{x+3} = \lim_{x\to -3^-} \frac{(x-2)(x+3)}{x+3} = \lim_{x\to -3^-}(x-2) = -5$$

$\blacksquare$

(d) $\displaystyle\lim_{x\to -3^-} \frac{x^2-x+6}{x+3}$

Solution: The denominator approaches zero as x approaches -3, but the numerator does not. Therefore this expression is unbounded near -3. When $x < -3$, the denominator is negative. The numerator is x^2-x+6 and at $x=-3$ has the value $18>0$. Therefore $\displaystyle\lim_{x\to -3^-} \frac{x^2-x+6}{x+3} = -\infty$. $\blacksquare$

(e) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+7x-4} - x - 3$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+7x-4} - x - 3 &= \lim_{x\to\infty} \frac{(\sqrt{x^2+7x-4}-(x+3))(\sqrt{x^2+7x-4}+(x+3))}{\sqrt{x^2+7x-4}+(x+3)} \\
&= \lim_{x\to\infty} \frac{(x^2+7x-4)-(x+3)^2}{\sqrt{x^2+7x-4}+(x+3)} \\
&= \lim_{x\to\infty} \frac{x-13}{\sqrt{x^2+7x-4}+(x+3)} \\
&= \lim_{x\to\infty} \frac{1-\frac{13}{x}}{\sqrt{1+\frac{7}{x}-\frac{4}{x^2}}+(1+\frac{3}{x})} \quad = \quad \frac{1}{2}
\end{aligned}$$

∎

(f) $\lim_{x\to-\infty} \sqrt{x^2+7x-4} - x - 3$

Solution: The square root expression becomes very large positive, and if $x \to -\infty$, $-x-3$ also becomes very large positive. Therefore the limit does not exist: $\lim_{x\to-\infty} \sqrt{x^2+7x-4} - x - 3 = \infty$. ∎

Question 2.

(a) Is the following function continuous at $x = 1$? at $x = 3$? If there are discontinuities at either of these points, describe them as infinite, jump, or removable as the case may be.

$$f(x) = \begin{cases} x & x < 1 \\ 2 & x = 1 \\ 2-x & 1 < x < 3 \\ 1 & x = 3 \\ x-2 & x > 3 \end{cases}$$

Note that there was a typographical error in the last line of this problem ($x > 2$ instead of $x > 3$)).

Solution: Since the function is defined by cases around the points in question, we have to work out the solution by cases as well; that is, we have to consider left-hand and right-hand limits.

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 2-x = 1$$

Therefore, $\lim_{x\to1} f(x)$ exists and equals 1. Since $f(1) = 2$, f is not continuous at $x = 1$; however the discontinuity is removable.

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} 2-x = -1$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} x-2 = 1$$

Therefore, $\lim_{x\to3} f(x)$ does not exist since the right-hand and left hand limits are different; however since both the right-hand and left hand limits exist, there is a jump discontinuity at $x = 3$. ∎

(b) Find a and b so that $g(x)$ is continuous everywhere.

$$g(x) = \begin{cases} 4-x^2 & x \le -1 \\ ax+b & -1 \le x < 2 \\ x^2-x-8 & 2 \le x \end{cases}$$

Solution: The function g is defined by cases. Within each of the open intervals $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$, g is defined by a single polynomial function and is therefore continuous within each of those intervals. At the points $x = -1$ and $x = 2$, we have to evaluate limits by taking left-hand and right-hand limits.

$$\lim_{x\to-1^-} g(x) = \lim_{x\to-1^-} 4-x^2 = 3 \qquad \lim_{x\to-1^+} g(x) = \lim_{x\to-1^+} ax+b = -a+b$$

$$\lim_{x\to2^-} g(x) = \lim_{x\to2^-} ax+b = 2a+b \qquad \lim_{x\to2^+} g(x) = \lim_{x\to2^+} x^2-x-8 = -6$$

For continuity, at each point we must have the left-hand limit equal to the right-hand limit, and equal to the value of the function at that point.

So we need $-a+b = 3$ $(= g(-1))$ and $2a+b = -6$ $(= g(2))$. From this it follows easily that $a = -3$ and $b = 0$ (and so $g(-1) = 3$ and $g(2) = -6$). ∎

Question 3. Find the derivative of $f(x)$ using the definition of the derivative.

(a) $f(x) = \dfrac{1}{\sqrt{2-x}}$

Solution: Note that $f(x)$ is only defined for $x \leq 2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{2-(x+h)}} - \dfrac{1}{\sqrt{2-x}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2-x} - \sqrt{2-(x+h)}}{h\sqrt{2-(x+h)}\sqrt{2-x}} \\
&= \lim_{h\to 0} \frac{(\sqrt{2-x} - \sqrt{2-(x+h)})(\sqrt{2-x} + \sqrt{2-(x+h)})}{h\sqrt{2-(x+h)}\sqrt{2-x}(\sqrt{2-x} + \sqrt{2-(x+h)})} \\
&= \lim_{h\to 0} \frac{(2-x)-(2-(x+h))}{h\sqrt{2-(x+h)}\sqrt{2-x}(\sqrt{2-x} + \sqrt{2-(x+h)})} \\
&= \lim_{h\to 0} \frac{h}{h\sqrt{2-(x+h)}\sqrt{2-x}(\sqrt{2-x} + \sqrt{2-(x+h)})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{2-(x+h)}\sqrt{2-x}(\sqrt{2-x} + \sqrt{2-(x+h)})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{2-x}\sqrt{2-x}(\sqrt{2-x} + \sqrt{2-x})} = \frac{1}{2(\sqrt{(2-x)^3}} = \frac{1}{2}(2-x)^{-3/2}
\end{aligned}$$

Note that there are several equivalent forms for writing the final answer; and that $f'(x)$ is defined only for $x < 2$. ∎

(b) $f(x) = \dfrac{-2}{x^2+4}$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{-2}{(x+h)^2+4} - \dfrac{-2}{x^2+4}}{h} \\
&= \lim_{h\to 0} \frac{-2\left((x^2+4)-((x+h)^2+4)\right)}{h((x+h)^2+4)(x^2+4)} \\
&= \lim_{h\to 0} \frac{-2\left(-2xh-h^2\right)}{h((x+h)^2+4)(x^2+4)} \\
&= \lim_{h\to 0} \frac{2(2x-h)}{((x+h)^2+4)(x^2+4)} \\
&= \frac{4x}{(x^2+4)^2}
\end{aligned}$$

∎