UNIVERSITY OF MANITOBA
Term Test 1B
COURSE: MATH 1500
DATE & TIME: October 9, 2018, 5:40PM – 6:40PM
CRN: various
DURATION: 1 hour EXAMINER: various

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 6 pages of questions and two blank pages for rough work. Please check that you have all the pages. **DO NOT REMOVE THE SCRAP PAPER**

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. Unjustified answers will receive little or no credit. **Do not continue on the back of the page.** If you need more space, continue on one of the scrap pages, CLEARLY INDICATING THAT YOUR WORK IS TO BE CONTINUED.

V. Do not deface the QR - code in the top right corner. Doing so may result in the page not being scanned and therefore not graded.

Question	Points	Score
1	20	
2	7	
3	7	
4	5	
5	5	
6	6	
Total:	50	

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1. Calculate each limit below, if it exists. If a limit does not exist, explain why. Show all work. Writing an answer with no justification may not yield any marks.

[3] (a) $\lim_{x\to 2^+} \dfrac{2-3x^2}{|x-2|}$.

Solution: $\lim_{x\to 2^+} 2-3x^2 = -10$ and $\lim_{x\to 2^+} |x-2| = \lim_{x\to 2^+} (x-2) = 0^+$.
Therefore the limit goes to either $\pm\infty$.
Since the fraction is negative, $\lim_{x\to 2^+} f(x) = -\infty$.

[5] (b) $\lim_{x\to 3} \dfrac{x-3}{\sqrt{2x+3}-\sqrt{3x}}$.

Solution: We rationalize the denominator:

$$\begin{aligned}
\lim_{x\to 3} \frac{x-3}{\sqrt{2x+3}-\sqrt{3x}} &= \lim_{x\to 3} \frac{(x-3)(\sqrt{2x+3}+\sqrt{3x})}{(\sqrt{2x+3}-\sqrt{3x})(\sqrt{2x+3}+\sqrt{3x})} \\
&= \lim_{x\to 3} \frac{(x-3)(\sqrt{2x+3}+\sqrt{3x})}{2x+3-3x} \\
&= \lim_{x\to 3} \frac{(x-3)(\sqrt{2x+3}+\sqrt{3x})}{-(x-3)} \\
&= \lim_{x\to 3} -(\sqrt{2x+3}+\sqrt{3x}) \\
&= -6.
\end{aligned}$$

[6] (c) $\lim_{x\to -\infty} \dfrac{\sqrt{9x^2+x}}{2x-1}$.

Solution: Notice that $\sqrt{x^2} = -x$ because $x < 0$.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{\sqrt{9x^2+x}}{2x-1} &= \lim_{x\to -\infty} \frac{\sqrt{x^2(9+\frac{1}{x})}}{2x-1} = \lim_{x\to -\infty} \frac{\sqrt{x^2}\sqrt{(9+\frac{1}{x})}}{x(2-\frac{1}{x})} \\
&= \lim_{x\to -\infty} \frac{-x\sqrt{(9+\frac{1}{x})}}{x(2-\frac{1}{x})} = \lim_{x\to -\infty} \frac{-\sqrt{(9+\frac{1}{x})}}{(2-\frac{1}{x})} \\
&= \frac{-\sqrt{(9+0)}}{(2-0)} = -\frac{3}{2}.
\end{aligned}$$

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[2] (d) $\lim_{x\to 3} \frac{x^3 - 3x^2}{x^2 - 9}$.

Solution:

$$\lim_{x\to 3} \frac{x^3 - 3x^2}{x^2 - 9} = \lim_{x\to 3} \frac{x^2(x-3)}{(x+3)(x-3)} = \lim_{x\to 3} \frac{x^2}{(x+3)} = \frac{3}{2}$$

[4] (e) $\lim_{x\to 1} (x-1)^2 \cos\left(\frac{2\pi}{\sqrt[3]{x-1}}\right)$.

Solution: For all $x \neq 1$, $-1 \leq \cos\left(\frac{2\pi}{\sqrt[3]{x-1}}\right) \leq 1$.

Thus, for $x \neq 1$,

$$-(x-1)^2 \leq (x-1)^2 \cos\left(\frac{2\pi}{\sqrt[3]{x-1}}\right) \leq (x-1)^2.$$

Since both $(x-1)^2 \to 0$ and $-(x-1)^2 \to 0$, as $x \to 1$, then by the Squeeze Theorem,

$$\lim_{x\to 1} (x-1)^2 \cos\left(\frac{2\pi}{\sqrt[3]{x-1}}\right) = 0.$$

[7] 2. Let f be the function:

$$f(x) = \begin{cases} c^2 x & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 3cx - 2 & \text{if } x > 1 \end{cases}.$$

Find all values of c for which $f(x)$ is continuous for all real numbers. Be sure to fully justify your answer.

Solution: For any c, the function is continuous on $(-\infty, 1) \cup (1, \infty)$ because it is a polynomial in those intervals.
The function will be continuous at $x = 1$ if

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 4.$$

Furthermore,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} c^2 x = c^2. \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3cx - 2 = 3c - 2.$$

Then,

$$c^2 = 4 = 3c - 2.$$

We now find c:

$$c^2 = 4, \quad c = \pm\sqrt{4}, \quad c = \pm 2 \qquad\qquad 3c - 2 = 4, \quad 3c = 6, \quad c = 2.$$

The only value of c that makes both above statements true is $c = 2$. We conclude that for $c = 2$ the function $f(x)$ is continuous for all real numbers.

[7] 3. Use the **definition of derivative** to find $f'(x)$. No credit will be given for any other method.

$$f(x) = \frac{1}{4+x}$$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{4+(x+h)} - \dfrac{1}{4+x}}{h} \\
&= \lim_{h\to 0} \frac{4+x-(4+(x+h))}{h(4+(x+h))(4+x)} \\
&= \lim_{h\to 0} \frac{4+x-4-x-h}{h(4+x+h)(4+x)} \\
&= \lim_{h\to 0} \frac{-h}{h(4+x+h)(4+x)} \\
&= \lim_{h\to 0} \frac{-1}{(4+x+h)(4+x)} \\
&= -\frac{1}{(4+x)^2}.
\end{aligned}$$

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[5] 4. Find the domain of the function $f(x) = \sqrt{\frac{2(x+1)}{(x^2-4)}}$. Express your final answer in terms of intervals.

Solution: $f(x) = \sqrt{\frac{2(x+1)}{(x^2-4)}} = \sqrt{\frac{2(x+1)}{(x-2)(x+2)}}$.
The radicand needs to be non-negative under the square root and non-zero in the denominator. Let $h(x) = \frac{2(x+1)}{(x-2)(x+2)}$

x	$x < -2$	$-2 < x < -1$	$-1 < x < 2$	$x > 2$
$x+2$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$-$	$+$	$+$
$h(x)$	$-$	$+$	$-$	$+$

Therefore, $D_f = (-2, -1] \cup (2, \infty)$.

[5] 5. Use the Intermediate Value Theorem to show that $x^3 - 4x + \frac{1}{x} = 0$ has at least one solution on the interval $[1, 2]$.

Solution: Let $f(x) = x^3 - 4x + \frac{1}{x}$, then $f(x)$ is continuous for all x except $x = 0$.
In particular, $f(x)$ is continuous on the interval $[1, 2]$.
We now check the value of $f(x)$ at the end points:

$$f(1) = (1)^3 - 4(1) + \frac{1}{1} = -2 < 0, \quad f(2) = 2^3 - 4(2) + \frac{1}{2} = \frac{1}{2} > 0$$

Since the values of the function at the end points have different signs, using the Intermediate Value Theorem, we conclude that there is a point c on the interval $[1, 2]$ such that $f(c) = 0$. Therefore, there exist at least one solution of the equation on $[1, 2]$.

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[6] 6. Use the **definition of derivative** to show that

$$f(x) = \begin{cases} x^2+1 & \text{if } x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

Solution:
The right-hand limit of the difference quotient of f at $x = 1$ is given by

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[2(1+h)] - 2}{h} = \lim_{h \to 0^+} \frac{2h}{h} = 2.$$

The left-hand limit of the difference quotient of f at $x = 1$ is given by

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2+1] - 2}{h} = \lim_{h \to 0^-} (2+h) = 2.$$

Since the left-hand and right-hand limits of the difference quotient are the same at $x = 1$, the function f is differentiable there.

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