UNIVERSITY OF MANITOBA
Term Test 1A
COURSE: MATH 1500
DATE & TIME: October 9, 2018, 5:40PM – 6:40PM
CRN: various
DURATION: 1 hour EXAMINER: various

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 6 pages of questions and two blank pages for rough work. Please check that you have all the pages. **DO NOT REMOVE THE SCRAP PAPER**

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. Unjustified answers will receive little or no credit. **Do not continue on the back of the page.** If you need more space, continue on one of the scrap pages, CLEARLY INDICATING THAT YOUR WORK IS TO BE CONTINUED.

V. Do not deface the QR - code in the top right corner. Doing so may result in the page not being scanned and therefore not graded.

Question	Points	Score
1	20	
2	7	
3	7	
4	5	
5	5	
6	6	
Total:	50	

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1. Calculate each limit below, if it exists. If a limit does not exist, explain why. Show all work. Writing an answer with no justification may not yield any marks.

[3] (a) $\lim\limits_{x \to 1^-} \dfrac{2x^2 - 1}{|x - 1|}$.

Solution: $\lim\limits_{x\to 1^-} 2x^2 - 1 = 1$ and $\lim\limits_{x \to 1^-} |x - 1| = \lim\limits_{x\to 1^-} -(x-1) = 0^+$.
Therefore the limit goes to either $\pm\infty$.
Since the fraction is positive, $\lim\limits_{x\to 1^-} f(x) = \infty$.

[5] (b) $\lim\limits_{x \to 3} \dfrac{\sqrt{x+3} - \sqrt{2x}}{x - 3}$.

Solution: We rationalize the numerator:

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+3}-\sqrt{2x}}{x-3} &= \lim_{x\to 3} \frac{(\sqrt{x+3}-\sqrt{2x})(\sqrt{x+3}+\sqrt{2x})}{(x-3)(\sqrt{x+3}+\sqrt{2x})} \\
&= \lim_{x\to 3} \frac{x+3-2x}{(x-3)(\sqrt{x+3}+\sqrt{2x})} \\
&= \lim_{x\to 3} \frac{-(x-3)}{(x-3)(\sqrt{x+3}+\sqrt{2x})} \\
&= \lim_{x\to 3} \frac{-1}{\sqrt{x+3}+\sqrt{2x}} \\
&= -\frac{1}{2\sqrt{6}}.
\end{aligned}$$

[6] (c) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+3x}}{3x+1}$.

Solution: Notice that $\sqrt{x^2} = -x$ because $x < 0$.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+3x}}{3x+1} &= \lim_{x\to-\infty} \frac{\sqrt{x^2(1+\frac{3}{x})}}{3x+1} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{(1+\frac{3}{x})}}{x(3+\frac{1}{x})} \\
&= \lim_{x\to-\infty} \frac{-x\sqrt{(1+\frac{3}{x})}}{x(3+\frac{1}{x})} = \lim_{x\to-\infty} \frac{-\sqrt{(1+\frac{3}{x})}}{(3+\frac{1}{x})} \\
&= \frac{-\sqrt{(1+0)}}{(3+0)} = -\frac{1}{3}.
\end{aligned}$$

[2] (d) $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$.

Solution:

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{(x+1)}{(x+2)} = \frac{3}{4}$$

[4] (e) $\lim\limits_{x\to 2} (x-2)^4 \sin\left(\dfrac{3}{(x-2)^2}\right)$.

Solution: For all $x \neq 2$, $-1 \leq \sin\left(\dfrac{3}{(x-2)^2}\right) \leq 1$.

Thus, for $x \neq 2$,

$$-(x-2)^4 \leq (x-2)^4 \sin\left(\frac{3}{(x-2)^2}\right) \leq (x-2)^4.$$

Since both $(x-2)^4 \to 0$ and $-(x-2)^4 \to 0$, as $x \to 2$, then by the Squeeze Theorem,

$$\lim_{x\to 2} (x-2)^4 \sin\left(\frac{3}{(x-2)^2}\right) = 0.$$

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[7] 2. Let f be the function:

$$f(x) = \begin{cases} (3k-2)x & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ k^2 x & \text{if } x > 1 \end{cases}.$$

Find all values of k for which $f(x)$ is continuous for all real numbers. Be sure to fully justify your answer.

Solution: For any k, the function is continuous on $(-\infty, 1) \cup (1, \infty)$ because it is a polynomial in those intervals.
The function will be continuous at $x = 1$ if

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1.$$

Furthermore,

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (3k-2)x \\ &= 3k-2. \end{aligned} \qquad \begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} k^2 x \\ &= k^2. \end{aligned}$$

Then,

$$3k - 2 = 1 = k^2.$$

We now find k:

$$\begin{aligned} 3k - 2 &= 1 \\ 3k &= 3 \\ k &= 1 \end{aligned} \qquad \begin{aligned} k^2 &= 1 \\ k &= \pm\sqrt{1} \\ k &= \pm 1. \end{aligned}$$

The only value of k that makes both above statements true is $k = 1$. We conclude that for $k = 1$ the function $f(x)$ is continuous for all real numbers.

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[7] 3. Use the **definition of derivative** to find $f'(x)$. No credit will be given for any other method.

$$f(x) = \frac{1}{3-x}$$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{3-(x+h)} - \dfrac{1}{3-x}}{h} \\
&= \lim_{h\to 0} \frac{3-x-(3-(x+h))}{h(3-(x+h))(3-x)} \\
&= \lim_{h\to 0} \frac{3-x-3+x+h}{h(3-x-h)(3-x)} \\
&= \lim_{h\to 0} \frac{h}{h(3-x-h)(3-x)} \\
&= \lim_{h\to 0} \frac{1}{(3-x-h)(3-x)} \\
&= \frac{1}{(3-x)^2}.
\end{aligned}$$

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[5] 4. Find the domain of the function $f(x) = \sqrt{\dfrac{5(9-x^2)}{(x-9)}}$. Express your final answer in terms of intervals.

Solution: $f(x) = \sqrt{\dfrac{5(9-x^2)}{(x-9)}} = \sqrt{\dfrac{5(3-x)(3+x)}{(x-9)}}$.
The radicand needs to be non-negative under the square root and non-zero in the denominator. Let $h(x) = \dfrac{5(3-x)(3+x)}{(x-9)}$

x	$x < -3$	$-3 < x < 3$	$3 < x < 9$	$x > 9$
$3-x$	$+$	$+$	$-$	$-$
$3+x$	$-$	$+$	$+$	$+$
$x-9$	$-$	$-$	$-$	$+$
$h(x)$	$+$	$-$	$+$	$-$

Therefore, $D_f = (-\infty, -3] \cup [3, 9)$.

[5] 5. Use the Intermediate Value Theorem to show that $x^3 - x - \dfrac{1}{x+2} = 0$ has at least one solution on the interval $[0, 2]$.

Solution: Let $f(x) = x^3 - x - \dfrac{1}{x+2}$, then $f(x)$ is continuous for all x except $x = -2$.
In particular, $f(x)$ is continuous on the interval $[0, 2]$.
We now check the value of $f(x)$ at the end points:

$$f(0) = 0^3 - 0 - \frac{1}{0+2} = -\frac{1}{2} < 0, \quad f(2) = 2^3 - 2 - \frac{1}{2+2} = \frac{23}{4} > 0$$

Since the values of the function at the end points have different signs, using the Intermediate Value Theorem, we conclude that there is a point c on the interval $[0, 2]$ such that $f(c) = 0$. Therefore, there exist at least one solution of the equation on $[0, 2]$.

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[6] 6. Use the **definition of derivative** to show that

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x \le 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

is not differentiable at $x = 1$.

Solution:
The right-hand limit of the difference quotient of f at $x = 1$ is given by

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 2] - 3}{h} = \lim_{h \to 0^+} 1 = 1.$$

The left-hand limit of the difference quotient of f at $x = 1$ is given by

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 2] - 3}{h} = \lim_{h \to 0^-} (2 + h) = 2.$$

Since the left-hand and right-hand limits of the difference quotient differ at $x = 1$, the function f is not differentiable there.

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