The naive version of the Fundamental Theorem of Calculus

Let $F(t)$ be the cumulative area function of $f(x)$ on the interval $[a,\, b]$. Suppose that $f(x)$ is continuous.

(a) $F(t)$ is differentiable, and $F'(t) = f(t)$ on $[a,\, b]$.

That is, F is an antiderivative of f on $[a,\, b]$. Furthermore, $F(a) = 0$ and $F(b) = \mathbf{A}_a^b f(x)$.

(b) Let G be any antiderivative of $f(x)$ on $[a,\, b]$. Then

$$\mathbf{A}_a^b f(x) = G(b) - G(a)$$

Outline of proof of the Fundamental Theorem (1)

$$
\begin{aligned}
F'(t) &= \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}[\mathbf{A}_a^{t+h} f(x) - \mathbf{A}_a^{t} f(x)) \\
&= \lim_{h \to 0} \frac{1}{h}\mathbf{A}_t^{t+h} f(x))
\end{aligned}
$$

Outline of proof of the Fundamental Theorem (2)

For each $h > 0$, since $f(x)$ is continuous, it has a minimum value m_h and a maximum value M_h on $[t, t+h]$. Comparing areas of rectangles and the area under the curve:

$$
\begin{aligned}
m_h &\leq \quad f(x) \quad \leq M_h \\
hm_h &\leq \quad \mathbf{A}_t^{t+h} \quad \leq hM_h \\
m_h &\leq \frac{1}{h}\mathbf{A}_t^{t+h} \leq M_h
\end{aligned}
$$

Outline of proof of the Fundamental Theorem (3)

Since $f(x)$ is continuous,

$$\lim_{h \to 0} m_h = f(t) = \lim_{h \to 0} M_h$$

Therefore, by the Squeeze Theorem,

$$F'(t) = f(t)$$