What's missing?

We need to:

1. *Define* an operation $\mathbf{A}_a^b$ on functions.

2. Prove that this operation satisfies our four (and a little bit) intuitive ideas about area.

3. Prove that if $f(x)$ is continuous, then $\mathbf{A}_a^b f(x)$ is defined.

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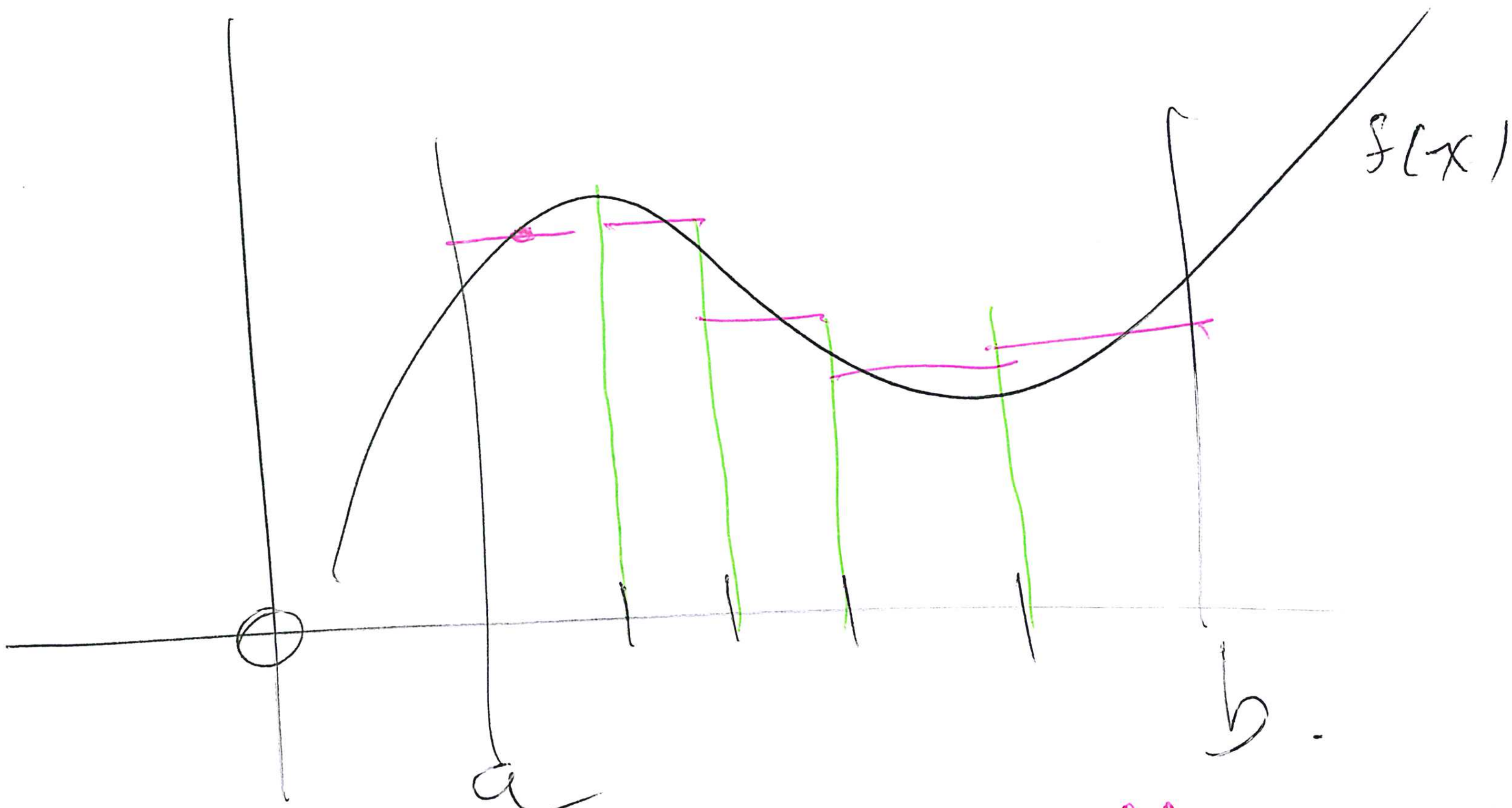
Approximating areas by rectangles.



Area ≃ the sum of the areas of the rectangles.

Partitions of an interval

A *partition* $\mathcal{P}$ of an interval $[a, b]$ is given by a sequence of points $a = x_0 < x_1 < \cdots < x_n = b$.

These points divide the interval $[a, b]$ into n *subintervals* . The *length* of the i-th subinterval is $\Delta x_i = x_i - x_{i-1}$.

The *norm* or *mesh* of $\mathcal{P}$ is $\|\mathcal{P}\|$, the length of the longest subinterval in $\mathcal{P}$: the maximum value amongst Δx_1 , Δx_2 ,. . . , Δx_n .

We concentrate almost exclusively on *uniform* or *regular* partitions: this is a partion $\mathcal{P}_n$ of $[a, b]$ into n equal subintervals, so $\|\mathcal{P}_n\| = \frac{b-a}{n}$, and $x_0 = a$, $x_1 = a + \frac{b-a}{n}$,. . . , $x_i = a + \frac{i(b-a)}{n}$, $x_n = b$.

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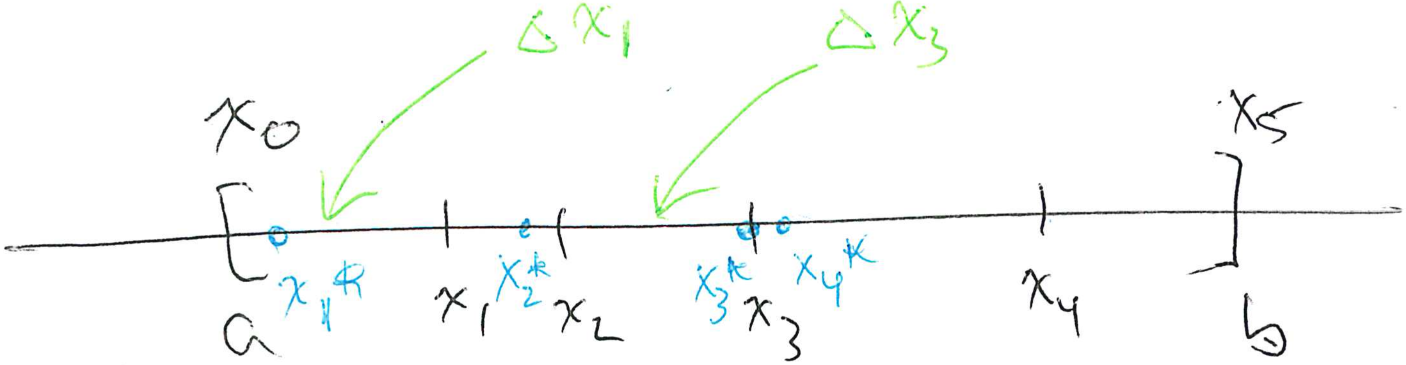

A regular partition is a partition into subintervals of the same length.

The length of the i-th subinterval is $\Delta x_i = x_i - x_{i-1}$

A regular partition of $[a,b]$ into n subintervals has $\Delta x_i = \frac{b-a}{n}$ for all i

Sample points for a partition

Let $\mathcal{P} = \{ x_0, x_1, \ldots, x_n \}$ be a partition of the interval $[a, b]$.

A *set of sample points* for this partition is any sequence

$$\mathcal{S} = x_1^*, x_2^*, \ldots, x_n^*$$

such that x_i^* is in the i-th subinterval: $x_{i-1} \le x_i^* \le x_i$.

For a good general theory, we need to allow completely random choices of sample points, but for the purposes of this course we can "cheat" a little.

insert picture here

Special sample points

There are several special ways of choosing sample points that can make our work a lot easier:

Left hand endpoints: Always choose $x_i^* = x_{i-1}$.

Right hand endpoints: Always choose $x_i^* = x_i$.

Midpoints: Choose $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$.

and if we are given a continuous function $f(x)$ on $[a, b]$:

Maximum values: Always choose x_i^* so that $f(x_i^*)$ is a maximum on $[x_{i-1}, x_i]$.

Minimum values: Always choose x_i^* so that $f(x_i^*)$ is a minimum on $[x_{i-1}, x_i]$.

Riemann sums

Given a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and a choice of sample points $\mathcal{S} = x_1^*, x_2^*, \ldots, x_n^*$ for this partition, and a function f defined on $[a, b]$, the associated *Riemann Sum* of f on $[a, b]$ is

$$\mathcal{R}(f, \mathcal{P}, \mathcal{S}) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Associated with our special choices of sample points are *left* Riemann sums, *right* Riemann sums, *upper* Riemann sums, and *lower* Riemann sums.

Definite integral

If we can force the Riemann sums of the function $f(x)$ on the interval $[a, b]$ to be as close to the number L as we want by taking the norm of the partition to be sufficiently small, then we say that the *definite integral* of $f(x)$ on the interval $[a, b]$ exists and equals L, and we write

$$\int_a^b f(x)\,dx = L$$