Four intuitive ideas about area

Area is always ≥ 0.

1. If a plane region is divided up into a finite number of parts by some curves, then the area of the region is the sum of the areas of its parts.

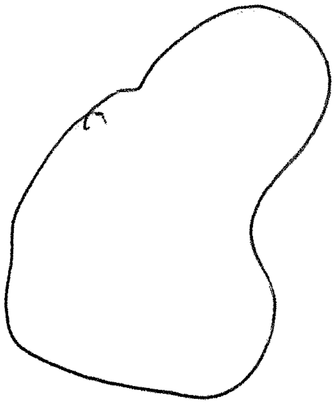
2. If one region is entirely contained inside another, then the area of the first region is no bigger than the area of the containing region.

3. If a plane region has no interior points, then its area is 0. Otherwise, its area (when defined) is positive.

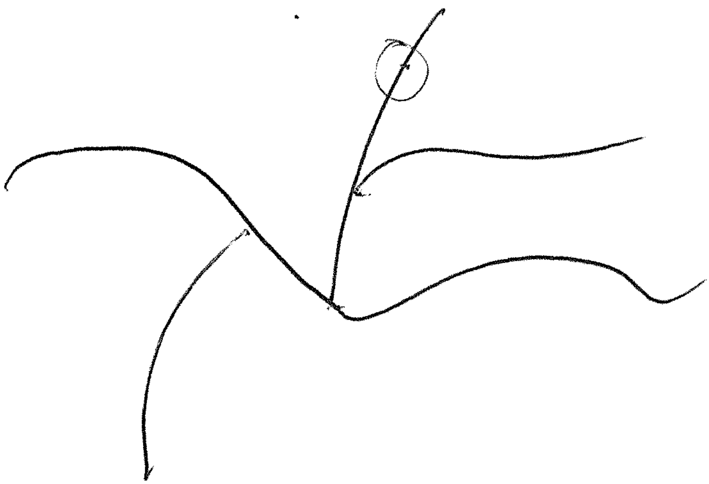
   [We call a point P an *interior point* of a region if some small enough circle around P lies entirely inside the region.]

4. The area of a rectangle is given by the formula

$$\text{area} = \text{base} \times \text{height}$$



probably
have area



area

Area as a calculus problem

The Standard Area Problem Given a reasonably well behaved function $y = f(x)$, **find** the area of the plane region *above* the x-axis, *below* the graph of $y = f(x)$, and *between* the vertical lines $x = a$ and $x = b$ (measuring from left to right).

Denote this area by $\mathrm{A}_a^b f(x)$.

Temporary notation

Note that *directions* are involved here: *above* the x-axis, from left to right, and so on.

If the "Standard Problem" is set up correctly, we expect to get a positive number for the area; but if we "measure backwards", we may get a negative number!

Geometric area

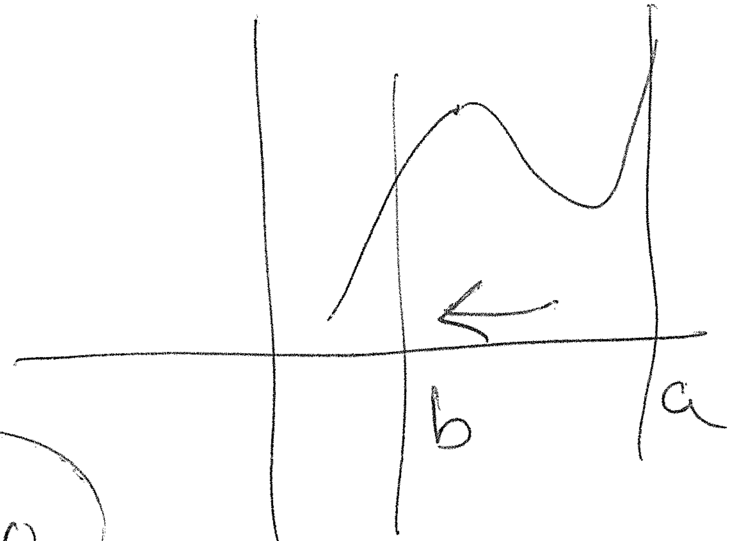
To calculate "true geometric area" we must impose additional conditions on the "Standard Problem":

1. We must have $a < b$.

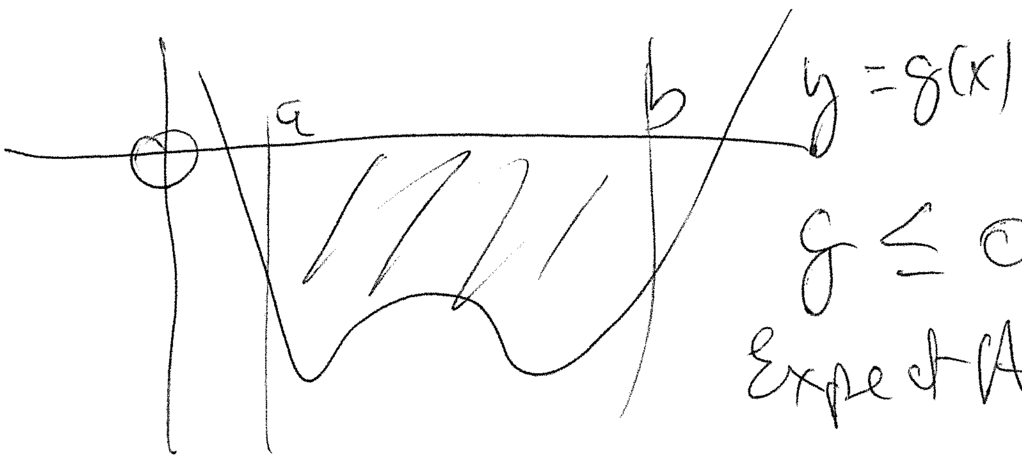
2. We must have $f(x) \geq 0$.

$A_a^b f(x) > 0$

Standard problem
geometric reconstruction



Standard problem
$b < a$
Measuring in negative direction
expect $A_a^b f(x) < 0$



$g \le 0$
Expect $A_a^b g(x) \le 0$

Four (and a little bit!) properties of Area

1. $\mathbf{A}_a^c f(x) = \mathbf{A}_a^b f(x) + \mathbf{A}_b^c f(x)$.

2. If $f(x) \leq g(x)$ for all x, $a \leq x \leq b$, then $\mathbf{A}_a^b f(x) \leq \mathbf{A}_a^b g(x)$.

3. If $f(a)$ is defined, then $\mathbf{A}_a^a f(x) = 0$.

4. If $f(x) = k$ for all x between a and b, then $\mathbf{A}_a^b f(x) = (b-a)k$.

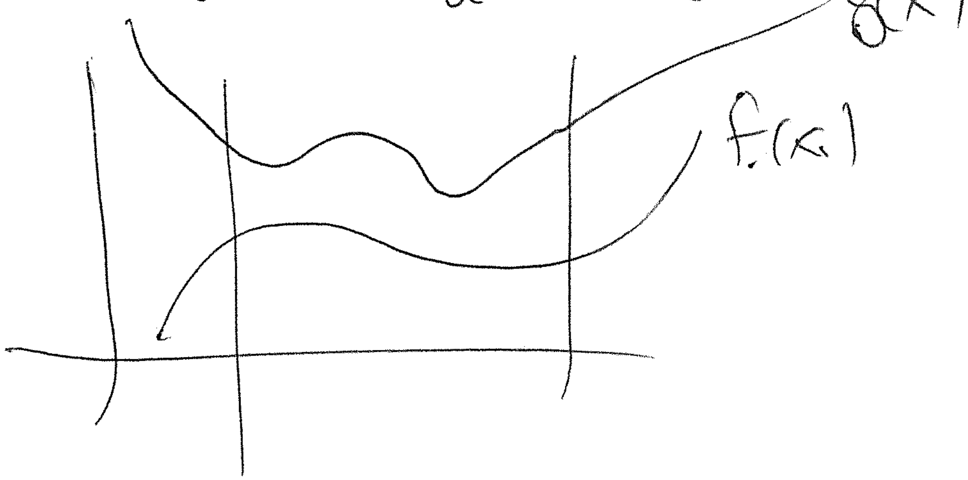
5. If $\mathbf{A}_a^b f(x)$ is defined and c is between a and b then $\mathbf{A}_a^c f(x)$ and $\mathbf{A}_c^b f(x)$ are both defined.

(1)

But we allow
a, b, c in any order!

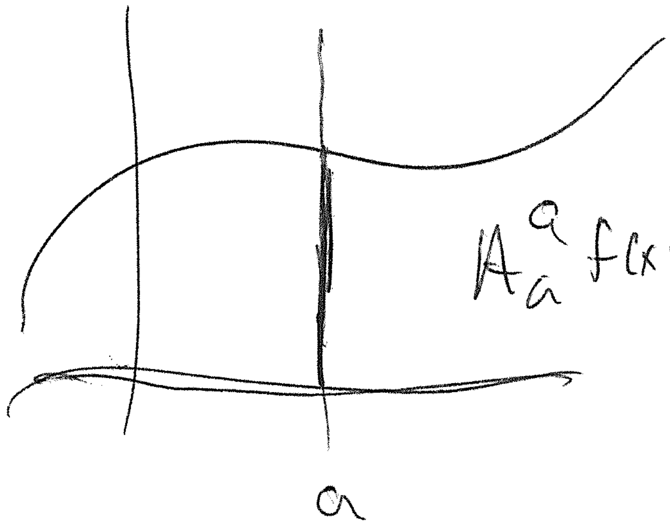
$$A_a^b f = A_a^c f + A_c^b f$$

(2)



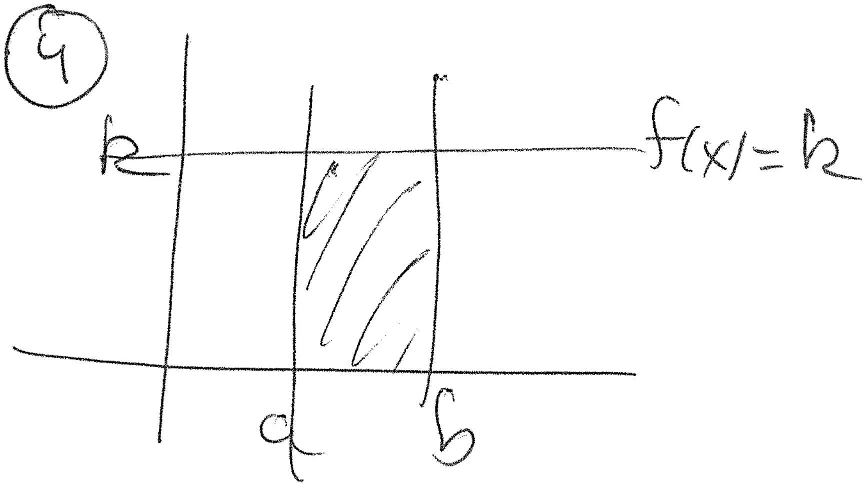
$$A_a^b f \le A_a^b g.$$

(3)



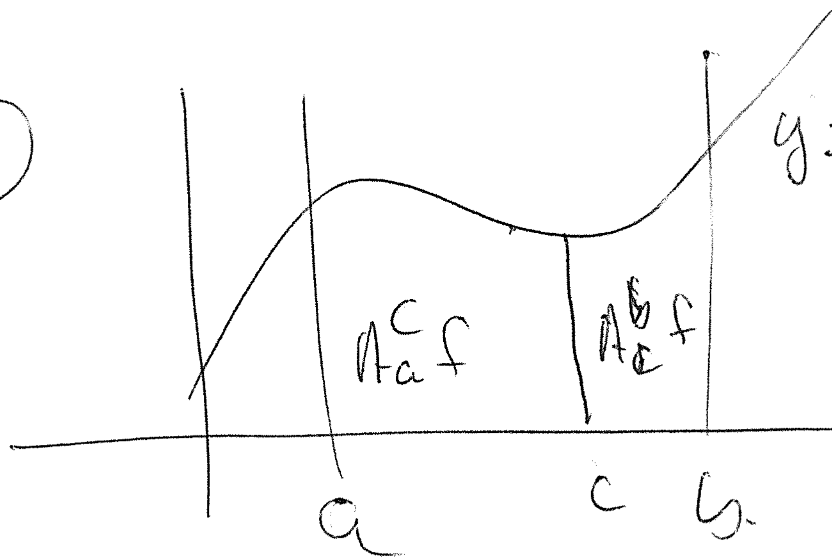
$A_a^a f(x) = 0$

(4)



$A_a^b f = (b-a)k$

(5)



$y = f(x)$. f is nice enough so that we can calculate $A_a^b f$

Then for any c, $a<c<b$, we can calculate the two pieces

Cumulative area functions

Supppose that $f(x)$ is a reasonably well-behaved function on $[a, b]$. $a < b$

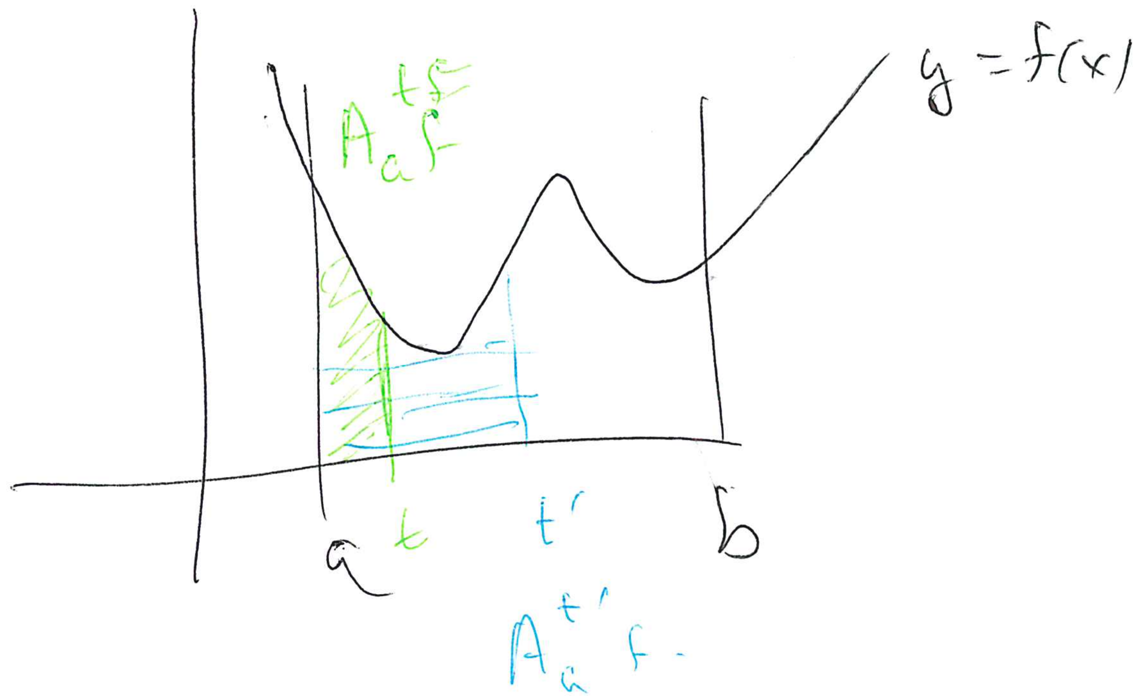
Define a new function $F(t)$,

the cumulative area function of $f(x)$

by

$$F(t) = \mathbf{A}_a^t f(x) \, ,$$

for all t in $[a, b]$ such that the area is defined.



$y = f(x)$
$A_a^t f$
a
t
t'
b
$A_a^{t'} f$

The naive version of the Fundamental Theorem of Calculus

Let $F(t)$ be the cumulative area function of $f(x)$ on the interval $[a, b]$. Suppose that $f(x)$ is continuous.

(a) $F(t)$ is differentiable, and $F'(t) = f(t)$ on $[a, b]$.

That is, F is an antiderivative of f on $[a, b]$. Furthermore, $F(a) = 0$ and $F(b) = \mathbf{A}_a^b f(x)$.

(b) Let G be any antiderivative of $f(x)$ on $[a, b]$. Then

$$\mathbf{A}_a^b f(x) = G(b) - G(a)$$