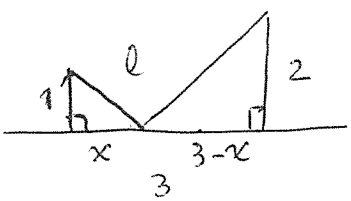
Problem: Two short poles, 1m and 2m high respectively, stand vertically on level ground 3m apart. A wire is supposed to run from the top of one pole to a point on the ground between the two poles and then to the top of the second pole. How far from the base of the 1m pole should the wire touch the ground so that the shortest possible length of wire is used?

Solution



Let x be the distance from the base of the 1m pole to the point where the wire touches the ground. Then $3-x$ is the remaining distance to the 2m pole. Let l be the total length of wire.

(Pythagorean theorem).

$$l = \sqrt{1+x^2} + \sqrt{(3-x)^2+4}$$

Problem: Find x so that l is a minimum, $0 \le x \le 3$.

$$\frac{dl}{dx} = \frac{1}{2\sqrt{1+x^2}}(2x) + \frac{1}{2\sqrt{(3-x)^2+4}}\,2(3-x)(-1) = \frac{x}{\sqrt{1+x^2}} - \frac{3-x}{\sqrt{(3-x)^2+4}}$$

No singular pts; critical points when $\dfrac{x}{\sqrt{1+x^2}} = \dfrac{3-x}{\sqrt{(3-x)^2+4}}$

$$\text{so } x^2((3-x)^2+4) = (1+x^2)(3-x)^2.$$

$$\cancel{9x^2} - \cancel{6x^3} + \cancel{x^4} + 4x^2 = 9 - 6x + x^2 + \cancel{9x^2} - \cancel{6x^3} + \cancel{x^4}$$

$$3x^2 + 6x - 9 = 0$$

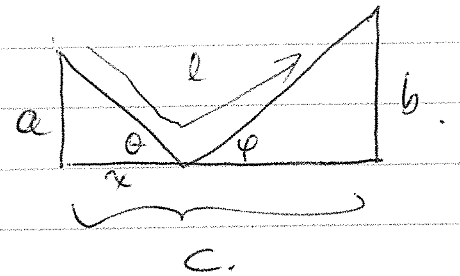
$$x^2 + 2x - 3 = 0; \quad (x+3)(x-1) = 0.$$

So $x = -3$ or $x = 1$; since $x \ge 0$ we only consider $x = 1$.

For $x < 1$, say $x = 0$, $\dfrac{dl}{dx} = -\dfrac{3}{\sqrt{13}} < 0$; For $x > 1$, say $x = 3$ $\dfrac{dl}{dx} = \dfrac{3}{\sqrt{10}} > 0$ so by the First Derivative Test, l has a local minimum at $x = 1$

The wire should touch the ground 1m from the 1m pole for the shortest length of wire to be used.

Notes on the ~~wire~~ "wire" problem



Minimize ℓ. $a<b$.

$$\ell = \sqrt{a^2+x^2} + \sqrt{(c-x)^2+b^2}$$

$$\frac{d\ell}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}} + \frac{\cancel{2}(c-x)(-1)}{\cancel{2}\sqrt{(c-x)^2+b^2}} = \frac{x}{\sqrt{a^2+x^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

$$\frac{d\ell}{dx} = 0 \text{ when } \frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{(c-x)^2+b^2}},$$ [hence when $\angle\theta = \angle\varphi$.]

Solving,

$$x^2((c-x)^2+b^2) = (c-x)^2(a^2+x^2.)$$

$$\cancel{c^2x^2} - \cancel{2cx^3} + \cancel{x^4} + b^2x^2 = a^2c^2 - 2a^2cx + a^2x^2 + \cancel{c^2x^2} - \cancel{2cx^3} + \cancel{x^4}$$

$$(b^2-a^2)x^2 + 2a^2cx - a^2c^2 = 0$$

Solving, $\Delta = 4a^4c^2 + 4(b^2-a^2)a^2c^2 = 4a^2b^2c^2$.

$$x = \frac{-2a^2c \pm 2abc}{b^2-a^2} = \frac{-ac}{b-a}, \frac{ac}{b+a}.$$

Only the 2^{nd} root is positive, and, $\frac{a}{a+b} < 1$, so $\frac{ac}{b+a} < c$.

Analyzing increase/decrease:

f': $+$ | $-$ | $+$

f: $\frac{-ac}{b-a}$ $\searrow$ 0 $\frac{ac}{b+a}$ $\nearrow$ c.

Clearly ~~$0 <$~~ $-\frac{ac}{b-a} < 0 < \frac{ac}{b+a} < c$; $f'(0) = \frac{-c}{\sqrt{c^2+b^2}} < 0$, $f'(c) = \frac{c}{\sqrt{a^2+c^2}} > 0$

$\therefore$ By first derivative test there is a local minimum at $x = \frac{ac}{b}$

When $x = \frac{ac}{a+b}$ the other distance is $c - \frac{ac}{a+b} = \frac{bc}{a+b}$

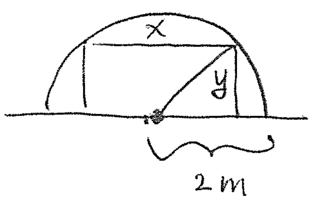
and this minimum length is

$$\ell = \sqrt{a^2 + \frac{a^2c^2}{(a+b)^2}} + \sqrt{\frac{b^2c^2}{(a+b)^2} + b^2} = \frac{a}{a+b}\sqrt{(a^2+b^2)+c^2} + \frac{b}{a+b}\sqrt{c^2+(a+b)}$$

$$= \sqrt{(a+b)^2+c^2}.$$

Problem: I have a semi-circular flower bed with a radius of 2m and the diameter of the semicircle against a fence. I have decided to organize a rectangular flower bed inside this semi-circular bed, with one edge of the rectangle along the fence as well. What is the largest possible area for such a flower bed?

Solution:



Let x be the length of the rectangle along the fence; let y be the perpendicular length. Let A be the area of the rectangle

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Clearly the largest rectangle touches the circumference of the semi-circle; so

$$\left(\frac{x}{2}\right)^2 + y^2 = 2^2$$

$$A = xy.$$

<u>Problem</u> find the maximum value of A, $0 \leq x \leq 4, 0 \leq y \leq 2$.

We get $y^2 = 4 - \left(\frac{x}{2}\right)^2$; $y = \sqrt{4 - \frac{x^2}{4}}$.

$\therefore\ A = x\sqrt{4 - \frac{x^2}{4}}$

$$\frac{dA}{dx} = 1 \cdot \sqrt{4 - \frac{x^2}{4}} + x \cdot \frac{1}{2\sqrt{4 - \frac{x^2}{4}}}\left(-\frac{1}{4}\,2x\right).$$

$$= \sqrt{4 - \frac{x^2}{4}} - \frac{x^2}{4\sqrt{4 - \frac{x^2}{4}}}.$$

$\frac{dA}{dx} = 0$ when $\sqrt{4 - x^2/4} = \frac{x^2}{4\sqrt{4 - x^2/4}}$ (singular point $x = 4$)

$$4\left(4 - \frac{x^2}{4}\right) = x^2.$$

$$16 = 2x^2; \quad x = \pm\sqrt{8}.$$

Since $0 \leq x \leq 4$, we only consider $x = \sqrt{8}$.

For $x < \sqrt{8}$, say $x = 2$, we get $\frac{dA}{dx} = \sqrt{3} - \frac{1}{\sqrt{3}} > 0$,

For $x > \sqrt{8}$, say $x = 3$, we get $\frac{dA}{dx} = \sqrt{7/4} - \frac{9}{4\sqrt{7/4}} = \frac{7-9}{4\sqrt{7/4}} < 0$

So by the first derivative test, A has a local max. at $x = \sqrt{8}$

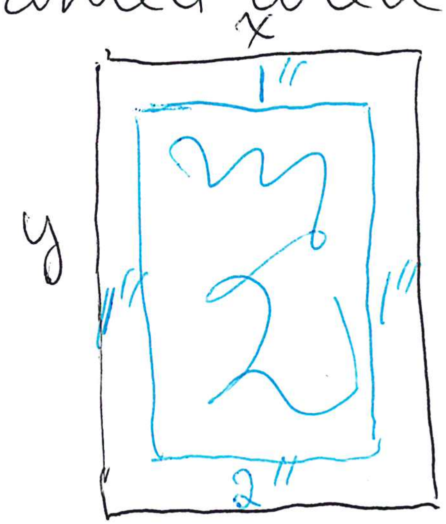
When $x = \sqrt{8}$, $A = \sqrt{8}\sqrt{4 - 8/4} = \sqrt{8}\sqrt{2} = 4$

The largest possible area for such a flower bed is $4\ m^2$.

11/21/5.

Problem (Set-up only)

We are going to design a ~~book page~~ poster with the following restrictions. The total area of the page will be 180 square inches. The printed image on the poster will be surrounded by margins of 1" on the top and sides and 2" on the bottom. Find the dimensions of the poster so that the printed area is as large as possible



Let x, y be the width, height of the poster

$$xy = 180, \quad x \geq 0, y \geq 0$$

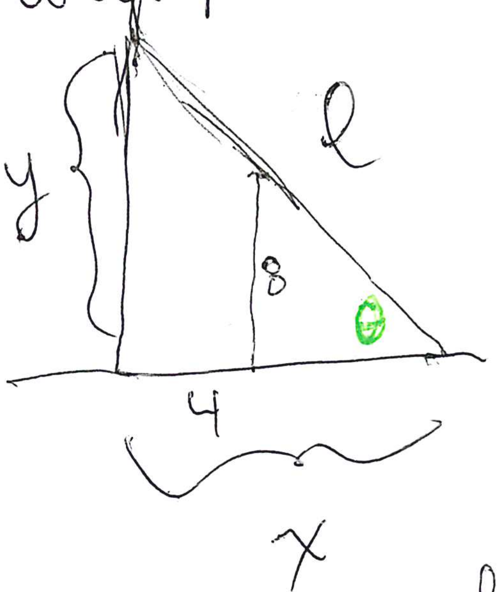
The dimensions of the printed area P are $x-2$ and $y-3$, so ~~P=~~

So $P = (x-2)(y-3)$, $x \geq 2, y \geq 3$

So $y = \frac{180}{x}$, so $P = (x-2)\left(\frac{180}{x} - 3\right) = 180 - \frac{360}{x} - 3x + 6$

Problem (Set up only) 11/21/6

A straight ladder leans against a ~~wall~~ vertical wall on level ground. It passed over an 8' fence, 4' from the wall. What is the shortest ladder that will pass over the fence from the wall to the ground?



Let l be the length of the ladder

Let x be the distance from wall to the base of the ladder

y the height of the top of the ladder against the wall

$$l^2 = x^2 + y^2 \qquad \frac{y}{x} = \frac{8}{4-x} \; ; \; y = \frac{8x}{4-x}$$

$$l^2 = x^2 + \frac{64x^2}{(4-x)^2}$$