Definite integral

If we can force the Riemann sums of the function $f(x)$ on the interval $[a, b]$ to be as close to the number L as we want by taking the norm of the partition to be sufficiently small, then we say that the *definite integral* of $f(x)$ on the interval $[a, b]$ exists and equals L, and we write

$$\int_a^b f(x)\,dx = L$$

REMINDER

§5.2 Continued

The Midpoint Rule

"Always choose the midpoints of a regular partition as sample points"

A good approximation to $\int_a^b f(x)\,dx$

is $\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$.

$(x_i = a + i\,\Delta x.)$.

Example 5 p384

Exercise 6 p 388

$\int_{-2}^{4} g(x)dx$, 6 subintervals.

The length of the domain is 6, so the partition is $x_0 = -2, -1, 0, 1, 2, 3, 4 = x_6$

In particular for rt hand endpoints, we estimate from the diagram

$$(-1.5)\cdot 1 + 0\cdot 1 + (1.5)\cdot 1 + (.5)\cdot 1 + (-1)\cdot 1 + (.5)1$$

$$= 0$$

left hand points:

$$0\cdot\underline{1} + (-1.5) + 0 + (1.5) + (.5) + (-1) \approx -.5$$

Statement of Fund. Thm

Properties of the definite integral (1)

Theorem: If f is continuous on $[a, b]$, or if f is defined on $[a, b]$ with only a finite number of jump discontinuities on $[a, b]$, then $\int_a^b f(x)\,dx$ exists.

Recall that f has a *jump discontinuity* at c in the domain of f if the left hand and right hand limits of f at c both exist but are different. [If c is an endpoint of the domain of f , we of course only require that the appropriate one-sided limit exist, and is different from $f(c)$.]

Properties of the definite integral (2)

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0 \quad \text{if } a \text{ is in the domain of } f$$

$$\int_a^b c\,dx = c(b-a) \quad \text{where } c \text{ is a constant}$$

$$\int_a^b (f(x) \pm g(x)) = \int_a^b f(x)\,dx \ \pm \ \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{where } c \text{ is a constant}$$

Properties of the definite integral (3)

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

The Fundamental Theorem of Calculus

Part I If f is continuous on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b) , and

$$F'(x) = f(x)\,.$$

So $F(x)$ is an antiderivative of $f(x)$, and such that $F(a) = 0$ and $F(b) = \mathrm{A}_a^b\, f(x)$.

Part II If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f on $[a, b]$.

Examples of questions about the fundamental theorem.

1) Functions defined by definite integrals — differentiation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

a) Suppose $F(x) = \int_0^x e^{\cos t}\,dt$. Find $F'(x)$

$$F'(x) = e^{\cos x}$$

Suppose $G(x) = \int_{-10}^x e^{\cos t}\,dt$. Find $G'(x)$.

$$G'(x) = e^{\cos x}$$

This should not be a surprise, because F and G differ by a constant;

$$G(x) = \int_{-10}^x e^{\cos t}\,dt = \int_{-10}^0 e^{\cos t}\,dt + \int_0^x e^{\cos t}\,dt$$

$$= (\text{constant}) + F(x).$$

(b) If $J(x) = \int_x^0 e^{\cos t}\,dt$, find $J'(x)$.

$J(x) = -\int_0^x e^{\cos t}\,dt$, so $J'(x) = -F'(x) = -e^{\cos x}$

(c) If $G(x) = \int_0^{x^2} e^{\cos t}\,dt$ find $G'(x)$.

Note that $G(x) = F(x^2)$

so by the chain rule, $G'(x) = F'(x^2)(2x)$

$G'(x) = e^{\cos(x^2)}(2x)$.

(d) If $H(x) = \int_{\sin x}^{0} e^{\cos(t)}\,dt$ find $H'(x)$

$H(x) = J(\sin x)$ so by the chain rule,

$H'(x) = J'(\sin x)\cos x$

$H'(x) = -e^{\cos(\sin x)}\cos(x)$

(e) If $K(x) = \int_{\sin x}^{x^2} e^{\cos t}\, dt$, find $\frac{d}{dx} K(x)$ 1130/5.

$$K(x) = \int_{\sin x}^{0} e^{\cos t}\, dt + \int_{0}^{x^2} e^{\cos t}\, dt$$

$$= H(x) + G(x)$$

So $K'(x) = H'(x) + G'(x)$

$$= -e^{\cos(\sin x)} \cos x + e^{\cos(x^2)}(2x)$$

$$= e^{\cos(x^2)}(2x) - e^{\cos(\sin x)} \cos(x)$$

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General patterns:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$\frac{d}{dx}\int_x^b f(t)\,dt = -f(x)$$

$$\frac{d}{dx}\int_a^{b(x)} f(t)\,dt = f(b(x))\,b'(x)$$

$$\frac{d}{dx}\int_{a(x)}^{b} f(t)\,dt = -f(a(x))\,a'(x)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x).$$