The University of Manitoba

Vector Geometry and Linear Algebra (MATH 1300)
Midterm Examination: Winter 2016

Time: 75 minutes Date: 25 February, 2016.

SOLUTIONS GUIDE

General comment: The INSTRUCTIONS included "You must **show your work** in order to get marks."

[9] **Question 1.** Consider the system of equations

$$x + a\,y = b, \quad 2\,x + y = 6,$$

in the variables x, y.

- Find all pairs (a, b) such that the system has a unique solution.
- Find all pairs (a, b) such that the system has an infinite number of solutions.
- Find all pairs (a, b) such that the system has no solution.

Comments: This question tests your understanding of how to classify the three different possibilities for the number of solutions of a system of linear equations. It is best done by finding the augmented matrix of the system. It turns out that one step of row-reduction is enough.

Solution: A system is *inconsistent* if it has a row-echelon form with a row where all the entries corresponding to variables are 0, but the constant is different from 0. If there is no such row, then the system must be consistent.

A consistent system has a *unique solution* if every variable corresponds to a "leading 1". A consistent system has *infinitely many solutions* if there are one or more "free variables" (variables which do not correspond to a "leading 1").

$$\left[\begin{array}{cc|c} 1 & a & b \\ 2 & 1 & 6 \end{array}\right] \quad R_2 - 2R_1 \qquad \left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1-2a & 6-2b \end{array}\right]$$

So there is a unique solution as long as $1 - 2a \neq 0$, that is, whenever $a \neq \frac{1}{2}$.

On the other hand, if $a = \frac{1}{2}$ then all the entries in the second row corresponding to variables are 0, and so:

The system has infinitely many solutions if $6 - 2b = 0$, that is, if $a = \frac{1}{2}$ and $b = 3$.

The system has no solution if $6 - 2b \neq 0$, that is, if $a = \frac{1}{2}$ and $b \neq 3$.

[8] **Question 2.** Find all pairs (p, q) such that matrix

$$\begin{bmatrix} 1 & p-q & 0 & 0 \\ 0 & p+2\,q & 1 & 0 \\ 0 & 0 & 0 & p+q \end{bmatrix}$$

is in reduced row-echelon form (RREF).

Comments: This question tests your understanding of the definition of *row-reduced echelon form*. It is a *fundamental error* to try to row-reduce the given matrix. It was a *common error* to assume that the entry in row 2, column 2 has to be zero, or that the entry in row 1, column 2 might be forced to be 1. It was a common oversight NOT to consider all of the possibilities. Among successful solutions, some of you preferred to start "from the left", others "from the right". A common source of errors was a poorly organized plan of attack.

Solution: Version 1: For the given matrix to be in row-reduced echelon form, the entry in row 2, column 2 must be 0 or 1 (and if it is 1, the entry above it must be 0; otherwise the entry in row 1, column 2 can be any number), AND the entry in row 3, column 4 must be 0 or 1. This gives (apparently) FOUR cases to consider.

If $p+2q=1$ then we also must have $p-q=0$, that is, $p=q$, and so $p+2p=1$, that is, $p=\frac{1}{3}$ and so $q=\frac{1}{3}$ as well. But then $p+q=\frac{2}{3}$, and so we cannot satisfy either condition on row 3, column 4.

If $p+2q=0$ then the entry above it can be any number, and so does not enter further into the solution. So $p=-2q$. Now $p+q=0$ or $p+q=1$, so $0=-2q+q=-q$ or $1=-2q+q=-q$. So we find (in the first case) that $(p,q)=(0,0)$ and (in the second case) that $(p,q)=(2,-1)$.

Solution: Version 2: For the given matrix to be in row-reduced echelon form, certainly the entry in row 3, column 4 can be either 0 or 1. In the first case, $q=-p$, and in the second case, $q=1-p$. Putting these values into the second column of the matrix, we find that it must have one of the following two forms:

$$\begin{bmatrix} 1 & 2p & 0 & 0 \\ 0 & -p & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2p-1 & 0 & 0 \\ 0 & 2-p & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now the entry in row 2, column 2 can be 0 or 1, and if is 1, the entry above it must be 0. We check both matrices, and if we make row 2, column 2 equal to 1, the entry in row 1, column 2 is not 1. If row 2, column 2 is 0, then (in the case of the first matrix) we find that $(p,q)=(0,0)$ and (in the case of the second matrix) we find that $(p,q)=(2,-1)$.

[10] **Question 3.** Let $A=\begin{bmatrix} 2 & -7 \\ -1 & 3 \end{bmatrix}$. Find the inverse of A using the method of row-operations. (No credit will be given for any other method.)

Comment: Set up the augmented matrix $[A \mid I]$ and row-reduce to $[I \mid A^{-1}]$, if possible. There are several equally valid pathways through the work, of which we show two.

A common ERROR was to fail to indicate which row operations you were using.

Solution: (Version 1)

$$\left[\begin{array}{rr|rr} 2 & -7 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{rr|rr} 1 & -\frac{7}{2} & \frac{1}{2} & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_2+R_1]{} \left[\begin{array}{rr|rr} 1 & -\frac{7}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[-2R_2]{}$$

$$\left[\begin{array}{rr|rr} 1 & -\frac{7}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_1+\frac{7}{2}R_2} \left[\begin{array}{rr|rr} 1 & 0 & -3 & -7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

Therefore $A^{-1}=\begin{bmatrix} -3 & -7 \\ -1 & -2 \end{bmatrix}$

Solution: (Version 2)

$$\left[\begin{array}{rr|rr} 2 & -7 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|rr} -1 & 3 & 0 & 1 \\ 2 & -7 & 1 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{rr|rr} 1 & -3 & 0 & -1 \\ 2 & -7 & 1 & 0 \end{array}\right] \xrightarrow[R_2-2R_1]{}$$

$$\left[\begin{array}{rr|rr} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow[-R_2]{} \left[\begin{array}{rr|rr} 1 & -3 & 0 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{rr|rr} 1 & 0 & -3 & -7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

[6] **Question 4.** . Find all 2×2 diagonal matrices A such that

$$A^2-A=2\,I.$$

Comment: You must know what a diagonal matrix is!

Solution: Since A is a 2×2 diagonal matrix, for some constants a and b, $A=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

Then $A^2-A=\begin{bmatrix} a^2-a & 0 \\ 0 & b^2-b \end{bmatrix}$ and so $a^2-a=2$ and $b^2-b=2$. Solving $0=a^2-a-2=(a-2)(a+1)$, we find $a=2$ or $a=-1$, and similarly for b.

Therefore there are four solutions:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

[9] **Question 5.** Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Comments: This question tests your knowledge of basic facts about elementary row operations, elementary matrices, and their connection to the evaluation of determinants.

The *elementary matrix* corresponding to a given elementary row operation is obtained from the identity matrix of the appropriate size by performing that elementary row operation.

You must show your work in reasonable detail.

(1) Find an elementary matrix E_1 such that $E_1 A = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$.

Solution: The given matrix is obtained from A by interchanging row 1 and row 3, so E_1 is obtained from I_3 in the same way: $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(2) Find an elementary matrix E_2 such that $A = E_2 \begin{bmatrix} a & b & c \\ d & e & f \\ g-3d & h-3e & i-3f \end{bmatrix}$.

Solution: A can be obtained from the given matrix by adding three times row 2 to row 3, so E_2 is obtained from I_3 in the same way: $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

(3) If it is given that $\det(A) = -3$, then find the following determinant:

$$\begin{vmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{vmatrix}.$$

Comment: If you interchange two rows of a matrix, you change the sign of the determinant of the matrix. If you multiply a row of a matrix by a constant, you multiply the value of the determinant by the same constant. Adding a multiple of one row to another does not change the value of the determinant.

Solution: The matrix given in this part is obtained from A by subtracting row 2 from row 1, and then interchanging the result with row 3 (or, alternatively, by first interchanging row 1 and row 3, and then subtracting row 2 from the (new) row 3.

In either case, since $\det(A) = -3$ the determinant of the new matrix is $(-1)(-3) = 3$.

[9] **Question 6.** Let $A = \begin{bmatrix} -1 & 2 & 5 & 6 \\ 0 & 3 & 1 & 7 \\ -2 & 6 & 2 & -2 \\ 2 & -3 & 4 & 3 \end{bmatrix}$. Find the cofactor C_{32} of A.

Comment: The (i, j) cofactor of a square matrix A is $(-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the matrix obtained from A by deleting row i and column j.

Solution: $C_{3,2} = (-1)^{3+2} \det \begin{bmatrix} -1 & 5 & 6 \\ 0 & 1 & 7 \\ 2 & 4 & 3 \end{bmatrix}$.

You can evaluate the determinant by any method you want, although expansion by cofactors along column 1 or along row 2 are the easiest options.

$$C_{3,2} = (-1)\left[(-1)\begin{vmatrix} 1 & 7 \\ 4 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 6 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & 6 \\ 1 & 7 \end{vmatrix}\right] =$$

$$= (-1)[(-1)(3-28) - 0 + 2(35-6)] = -[25+58] = -83$$

[9] **Question 7.** Let $A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 7 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ -2 & 0 \\ 5 & 2 \end{bmatrix}$.

Comment: There was some confusion between "trace" $\operatorname{tr}(A)$ and "transpose" A^T.

The *trace* of a square matrix is the sum of the diagonal entries. We have a theorem that (as long as both products are defined) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

For any $m \times n$ matrix A, A^T is the $n \times n$ matrix defined by $(A^T)_{i,j} = A_{j,i}$.

(1) Find $\operatorname{tr}(A\,B)$.

Solution: $AB = \begin{bmatrix} 2\cdot(-3)+(-1)\cdot(-2)+5\cdot 5 & * \\ * & 3\cdot 1+7\cdot 0+(-1)\cdot 2 \end{bmatrix} = \begin{bmatrix} 21 & * \\ * & 1 \end{bmatrix}$,
where stars represent some numbers which we do not need for computing the trace.

Hence, $\operatorname{tr}(A\,B) = 21 + 1 = 22$.

(2) Find $\operatorname{tr}(B\,A)$.

Solution:

Since we have a **Theorem** that for any matrices $\operatorname{tr}(B\,A) = \operatorname{tr}(A\,B)$, we immediately get that $\operatorname{tr}(B\,A) = 22$.

Alternatively $BA = \begin{bmatrix} (-3)\cdot 2+1\cdot 3 & * & * \\ * & (-2)\cdot(-1)+0\cdot 7 & * \\ * & * & 5\cdot 5+2\cdot(-1) \end{bmatrix} = \begin{bmatrix} -3 & * & * \\ * & 2 & * \\ * & * & 23 \end{bmatrix}$,
so $\operatorname{tr}(B\,A) = 22$.

(3) Let C be a matrix such that $A\,C\,B^T$ is defined. What is the size of C?

Solution: In order for $A\,C$ to be defined, the height of C has to be equal to the width of A, which is 3. In order for $C\,B^T$ to be defined, the width of C has to be equal to the height of B^T, which is equal to the width of B, which is 3. Thus, C has to be a 3×2 matrix.