UNIVERSITY OF MANITOBA

DATE: April 22, 2016 | Final Examination
TIME: 9:00am – 11:00am | DURATION: 2 hours
EXAMINATION: MATH 1300 |
CRN: 20395 | EXAMINERS: Various

Name: (print clearly) SOLUTIONS

Student number: DRAFT: Final version will be posted after grades have been posted to Aurora

I understand that cheating is a serious offence ____________ (Signature – *In Ink*)

Please place a check mark beside your section number and instructor:

- ☐ A01 — J. Chipalkatti (M/W/F 10:30–11:20am, Armes 200)
- ☐ A02 — T. Kucera (Tu/Th 10:00–11:15am, St. Paul's 100)
- ☐ A03 — I. Bilokopytov (M/W/F 1:30–2:20pm, Human Ecology 206)

INSTRUCTIONS

I. No texts or notes or other reference materials are permitted. No calculators, cellphones, electronic translators, or any WiFi-enabled devices are permitted.

II. This exam has a title page, 12 pages of questions and also one blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 120 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

V. For Questions 2 through 10, always show all your work with full explanations.

VI. After the exam is over, a full solution set will be posted on Prof. Kucera's website, which you can find by navigating from the Mathematics Department web site.

Question	Points	Score
1	20	
2	12	
3	12	
4	12	
5	8	
6	12	
7	12	
8	12	
9	8	
10	12	
Total:	120	

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1. Answer the following short-answer questions in the spaces provided. Write a Definition, state a Theorem, or do a short computation. Detailed explanations are *not* needed here.

[2] (a) A row echelon form of the system $A\mathbf{x} = \mathbf{0}$, where A is 7×10, has 4 non-zero rows. How many free variables (parameters) does the general solution to this system have?

> **Solution:** There are 10 variables and 4 "leading ones", so there are $10 - 4 = 6$ free variables (parameters).

[2] (b) What is the elementary matrix corresponding to the following elementary row operation?

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}$$

> **Solution:** It was "Row 1 replaced by Row 1 plus Row 2", so: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

[2] (c) Assume that A, B are matrices such that $B^T A^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find AB.

> **Solution:** $B^T A^T = (AB)^T$, so $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$.

[2] (d) Suppose that T is a linear operator $\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y - 2x \\ 3x + 2y \end{bmatrix}$. Find the matrix A so that for all $\mathbf{x} \in \mathbb{R}^2$, $T(\mathbf{x}) = A\mathbf{x}$.

> **Solution:** $\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

[2] (e) State the Cauchy-Schwarz inequality.

> **Solution:** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.

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[2] (f) Find the distance between the point $(2, 3)$ and the line $4x - 3y - 1 = 0$ in $\mathbb{R}^2$.

Solution: The general formula for the distance from (x_0, y_0) to the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}},$$

so in this case the distance is

$$D = \frac{|4 \cdot 2 - 3 \cdot 3 - 1||}{\sqrt{4^2 + 3^2}} = \frac{2}{5}.$$

[2] (g) State the formula for the orthogonal projection of a vector $\mathbf{u}$ on a non-zero vector $\mathbf{a}$.

Solution:

$$\text{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

[2] (h) Write the matrix of the linear operator on $\mathbb{R}^2$ corresponding to a rotation through an angle of $\frac{\pi}{3}$.

Solution: $\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

[2] (i) Write the characteristic polynomial of the matrix $\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$.

Solution: $\left|\begin{bmatrix} \lambda - 3 & 2 \\ -2 & \lambda - 4 \end{bmatrix}\right| = (\lambda - 3)(\lambda - 4) + 4 = \lambda^2 - 7\lambda + 16.$

[2] (j) Define "A and B are *similar* matrices".

Solution: A and B are *similar* matrices if for some invertible matrix P, $B = P^{-1}AP$.

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[12] 2. Let $\mathbf{u} = (2, 0, 2)$, $\mathbf{v} = (0, 1, 1)$, and $\mathbf{w} = (4, -3, 2)$.

Find numbers x, y, z, such that $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = (1, -1, -2)$.

Solution: By comparing first coordinates, second coordinates, and third coordinates we see that this is a system of linear equations in the three variables x, y, z, with augmented matrix as given below. We solve by row-reduction.

$$\left[\begin{array}{ccr|r} 2 & 0 & 4 & 1 \\ 0 & 1 & -3 & -1 \\ 2 & 1 & 2 & -2 \end{array}\right] \qquad \begin{array}{l} \text{(first) } R_3 - R_1 \\ \text{(then) } \frac{1}{2}R_1 \end{array}$$

$$\left[\begin{array}{ccr|r} 1 & 0 & 2 & \frac{1}{2} \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -2 & -3 \end{array}\right] \qquad R_3 - R_2$$

$$\left[\begin{array}{ccr|r} 1 & 0 & 2 & 1/2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore $x = \dfrac{9}{2}$, $y = -7$, and $z = -2$.

3. Let $A = \begin{bmatrix} x^2+1 & -11 & x \\ 0 & x^2-3x & 43 \\ 0 & 0 & x^2-1 \end{bmatrix}$.

[4] (a) Find all real numbers x such that A is *not* invertible.

Solution: The matrix A is a triangular matrix, so it is invertible if and only if all the diagonal entries are non-zero. x^2+1 is never 0, so the matrix is not invertible when $x^2-3x=0$ or when $x^2-1=0$, that is, when $x=-1,\ 0,\ 1,$ or 3.

[4] (b) Find the value of the determinant of A^{-1} when $x=2$.

Solution: A is a triangular matrix so its determinant is the product of the diagonal entries; and when a matrix is invertible, the determinant of the inverse matrix is the inverse of the determinant.
So when $x=2$, $|A| = 5\cdot(-2)\cdot 3 = -30$, and so $|A^{-1}| = \dfrac{-1}{30}$.

[4] (c) What are the eigenvalues of A when $x=0$? In this case, is A diagonalizable?

Solution: A is a triangular matrix, so its eigenvalues are the diagonal entries, which when $x=0$ are $1,\ 0,$ and -1. Since the 3×3 matrix A has 3 distinct eigenvalues, it is diagonalizable.

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[12] 4. Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 7 & 1 & -2 \end{bmatrix}$. Using the adjoint formula for A^{-1}, find the entry in the 1st row and 2nd column of A^{-1}. [No credit will be given for any other method.]

Solution: If $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

So we need the value of $\det(A)$ and the (1, 2) entry in $\operatorname{adj}(A)$.

We find the value of $\det(A)$ by any method. For reference, we give three possibilities here:

"Basketweave":

$$\begin{aligned} \det(A) &= -1\cdot 0\cdot(-2) + 2\cdot 1\cdot 7 + 4\cdot 3\cdot 1 - (4\cdot 0\cdot 7) - (2\cdot 3\cdot(-2)) - ((-1)\cdot 1\cdot 1) \\ &= 39 \end{aligned}$$

"Expansion by cofactors" (either along row 2 or along column 2 is best; we show "along row 2")

$$\det(A) = (-3)\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 7 & 1 \end{vmatrix} = (-3)(-4-4) - 1(-1-14) = 39$$

"Row reduction"

$$\begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 7 & 1 & -2 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + 7R_1 \end{matrix} \qquad \begin{bmatrix} -1 & 2 & 4 \\ 0 & 6 & 13 \\ 0 & 15 & 26 \end{bmatrix}$$

Further row-reduction does not look attractive, but now expansion by cofactors down column one is easy:

$$\det(A) = -1\begin{vmatrix} 6 & 13 \\ 15 & 26 \end{vmatrix} = (-1)(156 - 195) = 39$$

The adjoint of A is the transpose of the matrix of cofactors of A, so we need the (2, 1) cofactor. This is $(-1)^{2+1}M_{2,1}$, where $M_{2,1}$ is the (2, 1) minor of A, the determinant of the matrix obtained by striking out row 2 and column 1 of A:

$$M_{2,1} = \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -4 - 4 = -8$$

Therefore the entry in row 1 and column 2 of A^{-1} is

$$\frac{1}{39}(-1)(-8) = \frac{8}{39}$$

[8] 5. Consider the system of equations

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 7 \end{aligned}$$

Find the value of x using Cramer's rule. [No credit will be given for any other method.]

Solution: By Cramer's rule, the value of x is a ratio of two determinants: the determinant of the coefficient matrix in the denominator, and the determinant of the matrix formed from the coefficient matrix by replacing the first column (the one corresponding to x) by the column of constants.

We compute:

$$\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1 - 2 = -3 \qquad \begin{vmatrix} 2 & 2 \\ 7 & -1 \end{vmatrix} = -2 - 14 = -16\,,$$

and so

$$x = \frac{16}{3}\,.$$

6. Consider the vectors

$$\mathbf{u} = (\,-1,\,2,\,5\,)\,, \qquad \mathbf{v} = (\,4,\,1,\,-3\,)$$

in $\mathbb{R}^3$. Let θ denote the angle between them.

[5] (a) Use the dot product of $\mathbf{u}$ and $\mathbf{v}$ to find $\cos\theta$. Is θ acute or obtuse?

Solution:

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

So we compute $\mathbf{u}\cdot\mathbf{v} = (-1)\cdot 4 + 2\cdot 1 + 5\cdot(-3) = -17$, $\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$, and $\|\mathbf{v}\| = \sqrt{(4)^2 + 1^2 + (-3)^2} = \sqrt{26}$. Thus

$$\cos(\theta) = \frac{-17}{\sqrt{30}\sqrt{26}}\,.$$

Since this is less than zero, the angle is obtuse.

[5] (b) Use the cross product of $\mathbf{u}$ and $\mathbf{v}$ to find $\sin\theta$.

Solution:

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta)\,.$$

So we compute

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \left(\begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix}, -\begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} \right) \\
&= (\,(2\cdot(-3) - 5\cdot 1),\ -((-1)\cdot(-3) - 5\cdot 4),\ ((-1)\cdot 1 - 2\cdot 4)\,) \\
&= (\,-11,\ 17,\ -9\,)
\end{aligned}$$

and then

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-11)^2 + 17^2 + 9^2} = \sqrt{491}\,.$$

We have already calculated the other two norms in part (a) so we find

$$\sqrt{491} = \sqrt{30}\sqrt{26}\sin(\theta)\,,$$

and therefore $\sin(\theta) = \dfrac{\sqrt{491}}{\sqrt{30}\sqrt{26}}$.

[2] (c) Verify that your answers to parts (a) and (b) are correct by checking that

$$\cos^2\theta + \sin^2\theta = 1\,.$$

Solution:

$$\cos^2\theta + \sin^2\theta = \frac{289}{30\cdot 26} + \frac{491}{30\cdot 26} = \frac{289+491}{780} = 1.$$

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[12] 7. Find an equation of the plane in $\mathbb{R}^3$ containing the points

$$A = (1, -1, 0), \quad B = (0, 0, -2), \quad C = (1, 1, 1).$$

Solution: There are many different forms for a correct answer. From the three given points, we can use any two of them to determine two direction vectors parallel to the plane; and any one of them to determine a point on the plane. So can choose any two of $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$ (or their opposites $\overrightarrow{BA}$, $\overrightarrow{CA}$, and $\overrightarrow{CB}$) as direction vectors.

So one possible solution is to take

$$\mathbf{v}_1 = \overrightarrow{AB} = (0, 0, -2) - (1, -1, 0) = (-1, 1, -2),$$

$$\mathbf{v}_2 = \overrightarrow{AC} = (1, 1, 1) - (1, -1, 0) = (0, 2, 1),$$

$$\mathbf{x}_0 = A = (1, -1, 0).$$

and so we find the following equation for the plane:

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

or in alternate forms

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

or

$$\begin{cases} x = 1 - t_1 \\ y = -1 + t_1 + 2t_2 \\ z = -2t_1 + t_2 \end{cases}$$

In general, for this form of the solution, $\mathbf{x}_1$ can be any one of A, B, or C, and $\mathbf{v}_1$ and $\mathbf{v}_2$ can be any two of

$$\pm(-1, 1, -2), \quad \pm(0, 2, 1), \quad \pm(1, 1, 3).$$

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[12] 8. Consider the vectors

$$\mathbf{u} = (\, k,\, 2,\, 1\,)\,, \qquad \mathbf{v} = (\, 3,\, -1,\, 0\,)\,, \qquad \mathbf{w} = (\, 5,\, 1,\, -2\,)$$

in $\mathbb{R}^3$.

Find **all** the values of k such that the volume of the parallelopiped generated by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is 10.

Solution: The volume is given by the absolute value of the determinant of the 3×3 matrix whose rows are the three vectors.

Any method of calculating the value of the determinant is acceptable. We will show "expansion by cofactors down the third column".

So the volume is

$$\begin{vmatrix} k & 2 & 1 \\ 3 & -1 & 0 \\ 5 & 1 & -2 \end{vmatrix} = 1 \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + (-2) \begin{vmatrix} k & 2 \\ 3 & -1 \end{vmatrix} = (3 - (-5)) - 2(-k - 6) = 2k + 20$$

Therefore for the volume to be 10, we must have $|2k + 20| = 10$, so $2k + 20 = 10$ and $k = -5$, or $2k + 20 = -10$ and $k = -15$.

[8] 9. $\mathbf{T}$ is a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 5 \\ -9 \end{bmatrix}, \quad \text{and } \mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -6 \\ 5 \end{bmatrix}.$$

(a) Find the standard matrix A of $\mathbf{T}$.

Solution: The columns of the standard matrix are the values of $\mathbf{T}$ at the standard unit vectors, so $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -6 \\ 4 & -9 & 5 \end{bmatrix}$.

(b) Evaluate $\mathbf{T}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, using part (a) or by any other method.

Solution: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, so the answer is $\mathbf{T}(\mathbf{e}_1) + \mathbf{T}(\mathbf{e}_2) + \mathbf{T}(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

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10. Consider the matrix

$$A = \begin{bmatrix} 22 & -10 \\ 50 & -23 \end{bmatrix},$$

We have chosen the entries carefully so that all the calculations in the following result in reasonably small numbers.

[8] (a) Find all the eigenvalues of A and an eigenvector associated with each.

Solution: To find the eigenvalues we solve the characteristic equation.

$$\begin{vmatrix} \lambda - 22 & 10 \\ -50 & \lambda + 23 \end{vmatrix} = (\lambda-22)(\lambda+23)+500 = \lambda^2+\lambda-506+500 = \lambda^2+\lambda-6 = 0$$

so

$$(\lambda + 3)(\lambda - 2) = 0$$

or $\lambda = -3$, $\lambda = 2$.

To find an eigenvector belonging to -3, solve $((-3)-22)x+10y = 0$ or $y = \frac{5}{2}x$. Thus one eigenvector is $(2, 5)$.

To find an eigenvector belonging to 2, solve $(2 - 22)x + 10y = 0$ or $y = 2x$. One eigenvector is $(1, 2)$.

[4] (b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. You *do not* need to calculate P^{-1} or verify that the equation holds.

Solution: P has as its columns the eigenvectors, and D has as its diagonal entries the corresponding eigenvalues. So

$$P = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvectors and eigenvalues can be written in reverse order; and any (non-zero) multiples of the given eigenvectors are correct answers.