Injective, Pure-injective, and Saturated Modules

A brief overview — Corrected and expanded, February 2024

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Injective modules

Conventions:

- R is a ring with unit;
- All modules are left R-modules;
- All ideals are left ideals;
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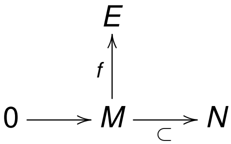
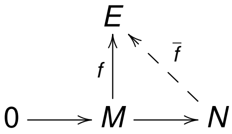
Injective Modules: Definition

A module $_RE$ is injective iff

Every diagram of left R-modules of the form:



can be completed as follows:



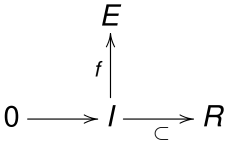
Injective modules: Standard examples

- Abelian groups $\mathbb{Q}$; $\mathbb{Z}(p^\infty)$ (p a prime); $\mathbb{R}/\mathbb{Z}$.
- Any (non-zero) vector space over a division ring.
- Any direct summand of an injective module;
- Any direct product of injective modules;
- Every non zero module is injective iff R is semisimple.

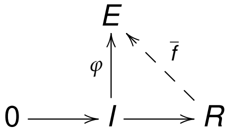
Injective modules: Baer's criterion

A module ${}_RE$ is injective iff

Every diagram of a left ideal of R of the form:



can be completed as follows:



Set Theory!

The proof requires the use of the Axiom of Choice.
In fact, the statement "$\mathbb{Q}$ is an injective abelian group" implies a fragment of Countable AC.

Systems of linear equations (1)

Let $\overline{x} = \langle x_j \rangle_{j \in J}$ be a (possibly infinite!) list of distinct variables and N a left R-module.

linear equation over N

A linear equation in $\overline{x}$ over N is just some finite linear combination

$$\sum_{j \in J} r_j x_j = n\,,$$

(that is, $r_j = 0$ for all but finitely many $j \in J$),
where each $r_j \in R$ and $n \in N$.

Systems of linear equations (2)

System of linear equations

A system of linear equations in $\overline{x}$ over N is just some set, possibly infinite, of such equations.

Matrix representation

Any such system can be represented in the form

$$A\overline{x} = \overline{b}$$

where A is a row-finite $I \times J$ matrix over R
and $\overline{b}$ is a $I \times 1$ column of constants from N.

Systems of linear equations: Satisfiability

Solutions

Suppose $N \leq M$.
A solution to $A\overline{x} = \overline{b}$ is some $\overline{m} \in M$ such that $A\overline{m} = \overline{b}$ (in M).

Satisfiability

$A\overline{x} = \overline{b}$ is satisfiable iff for some $M \geq N$, $A\overline{x} = \overline{b}$ has a solution in M.

Systems of linear equations: (formal) consistency I

Suppose $N \leq M$; $A\overline{x} = \overline{b}$ is a system of linear equations over N.

Consistency

$A\overline{x} = \overline{b}$ is (formally) consistent iff for all $(1 \times I)$ $\overline{r} \in R$,
($r_i = 0$ for all but finitely many $i \in I$),

$$\overline{r}A = 0 \text{ implies } \overline{r}\overline{b} = 0$$

Lemma

If $A\overline{x} = \overline{b}$ is satisfiable, then it is consistent.

Theorem

If $A\overline{x} = \overline{b}$ is consistent, then it is satisfiable.

Systems of linear equations: (formal) consistency II

For the Theorem:

- Let $\mathcal{E}$ be the submodule of $\overline{N} = R^{(J)} \oplus N$ generated by the rows of $\left[A| - \overline{b}\right]$.
- Let $M = \overline{N}/\mathcal{E}$.
  Then N embeds in M—follows from consistency!
- Let $\left\langle \underline{\mathrm{e}}_j \right\rangle_{j \in J}$ be the standard basis of the free module $R^{(J)}$;
- let $n_j = \underline{\mathrm{e}}_j / \mathcal{E}$.
- Then $\left\langle n_j \right\rangle j \in J$ is a solution of $A\overline{x} = \overline{b}$ in M.

Injectivity and systems of linear equations I

Theorem

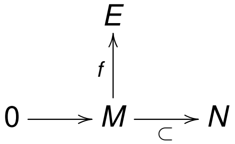
$_RE$ is injective iff every formally consistent system of linear equations over E has a solution in E.

Theorem (Baer's Criterion)

$_RE$ is injective iff every formally consistent system of linear equations in one variable over E has a solution in E.

Injectivity and systems of linear equations II

For the Theorem: the forward direction is easy; for the converse:
Given a diagram of left R-modules of the form:



- Let J enumerate N as $\overline{n}$;
- let $\langle x_j \rangle_{j \in J}$ be variables;
- Let $A\overline{x} = \overline{b}$ list all the equations $\overline{r} \cdot \overline{x} = b$, with constant $b \in M$, such that $\overline{n} \cdot \overline{x} = b$.

Then $A\overline{x} = f[\overline{n}]$ is consistent in E because f is a homomorphism, and a solution $\overline{x}$ in E defines a homomorphism $\bar{f} : N \to E$, because all the 'arithmetic' of N is encoded in the linear system.

Injective extensions

Theorem

Every module N can be embedded in an injective module.

"Proof"

The proof of the theorem "consistency implies satisfiability" shows how to add a solution to one consistent system of linear equations over N. A potentially VERY long construction by transfinite recursion will eventually give an injective extension.

Remark: I do have some notes on a probably mostly correct proof of this result.

$< (2^{|R|+|N|})^+$ is probably enough

Pure-injective modules

Purity (1)

positive primitive formulas

A positive primitive formula (ppf) is some existential quantification of a (finite) system of linear equations.

For instance, divisibility: $\exists u \, ru = x$.

matrix representation

$$\varphi(\overline{x}, \overline{b}) = \exists \overline{u} \, \left(A \begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} = \overline{b} \right)$$

where A is a (finite) matrix over R of the appropriate shape, $\overline{x}$ and $\overline{u}$ are finite lists of variables, and $\overline{b}$ are constants from some module N .

Purity (2): pure embeddings

Theorem

Let $M \leq N$ be R-modules.
The following are equivalent:

- *Every Σ (a finite system of linear equations over M) that has a solution in N has a solution in M.*
- *Every ppf $\varphi(\overline{x}, \overline{b})$ over M that has a solution in N has a solution in M.*

Definition

In this situation we say that M is pure in N, or that M is a pure submodule of N

Pure embeddings–examples

$$M \leq M \oplus N$$

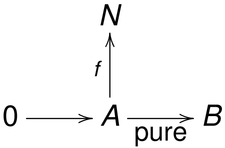
$$\mathbb{Z}_{(p)} \leq \overline{\mathbb{Z}_{(p)}}$$

NOT, for instance, $\mathbb{Z} \leq \mathbb{Q}$, or $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}$.

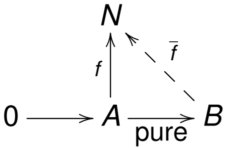
Pure-injectivity

A module ${}_RN$ is pure-injective iff it is injective over pure embeddings, that is

Every diagram of left R-modules of the form:



can be completed as follows:



Clearly every injective module is pure-injective.

Equational compactness

Definition

N is equationally compact iff for all systems of linear equations Σ over N, if every finite subset of Σ has a solution in N, then Σ has a solution in N.

Kaplansky [1954] and [1969] for abelian groups; Łoś et al. [1957]; Mycielski [1964, 1968] for universal algebra.

We say that "If Σ is finitely satisfiable, then it is satisfiable."

Standard exampled

If N carries a compatible compact Hausdorff topology, then N is equationally compact.

- e.g. any finite module.
- The abelian group $\mathbb{R}/\mathbb{Z}$.
- The p-adic group $\overline{\mathbb{Z}_{(p)}}$.

Pure-injectivity and equational compactness

Theorem

N is pure-injective
iff
N is equationally compact

NOTE: "Equational compactness" is a definition of universal algebra. It does not necessarily imply conditions similar to pure-injectivy in other kinds of algebras.

Saturated modules

A little bit of general Model Theory (1)

- an arbitrary first-order language—function symbols, constant symbols, relation symbols;
  - e.g. the usual first-order language for modules over a fixed ring R: $\langle +, -, 0, \underline{r}(-)\rangle_{r\in R}$, where each $\underline{r}$ is a unary operation representing scalar multiplication by r.
  - The usual language for ordered fields $\langle +, -, 0, \times, 1, <\rangle$.

- formulas formed from terms and relations between them; combined using all propositional connectives ($\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots$) and quantifiers $\forall, \exists$; and allowing parameters from some subset A of some structure;

- in particular, sentences are formulas with no free variables: they assert something that will be true or false about a structure as a whole, not about individual elements (tuples) of the structure.
  - $(\forall x)[(\underline{s}x \neq 0) \longrightarrow (\exists y)(\underline{r}y = x)]$.
  - $(\forall x)[(0 < x \longrightarrow (\exists y)(y \times y = x)]$.

A little bit of general Model Theory (2)

- A theory is a set of sentences, usually taken to be deductively closed and consistent (does not contain, e.g., $\exists x(x \neq x)$).
- Two structures for the same language are elementarily equivalent, $M \equiv N$, if they satisfy exactly the same sentences.
- If M is a substructure of N, then M is an elementary substructure of N, $M \preceq N$ if for some/any enumeration of M as $\overline{m}$, $\langle M, \overline{m} \rangle \equiv \langle N, \overline{m} \rangle$.

A little bit of general Model Theory (3): types

Let N be a structure, $A \subseteq N$, $\overline{b}$ some (usually finitely many) elements of N, and $\overline{x}$ a list of variables matching $\overline{n}$.

- The type of $\overline{b} \in N$ over A is the set of all such formulas in $\overline{x}$ and with parameters from A which are true in N of $\overline{b}$.

- Such a type is a complete, consistent set of formulas over A: for any such formula φ over A, either φ or $\neg\varphi$ is a property of $\overline{b}$, but not both.

- Conversely, if $p(\overline{x})$ is a complete, consistent set of formulas over A, there is $M \succeq N$ such that p is the type of some $\overline{b} \in M$ over A.

A little bit of general Model Theory (4): Saturation

Definition

Let $\kappa \geq |R|$ be an infinite cardinal.
N is κ-saturated if every type over every $A \subset N$, $|A| < \kappa$, is realized in N.

Clearly we must have $|N| \geq \kappa$.

Definition

N is saturated if it is $|N|$-saturated.

Basic facts

- It is consistent with ZFC that there are no uncountable saturated dense linear orders; in fact, the cardinality of such is a “large” cardinal.
- If there is a strongly inaccessible cardinal $\kappa > |T| + |N|$, N infinite, then N has a saturated elementary extension of cardinality κ.
  - (κ is regular and $\lambda < \kappa$ implies $2^\lambda < \kappa$.)
- If a theory T is stable in cardinality κ, then it has a saturated model of cardinality κ.
- Examples
  - Any vector space over a (skew) field K of dimension $\geq |K|$.
  - Any algebraically closed field of infinite transcendence degree.

Saturated modules

- Examples:
  - $\mathbb{R}/\mathbb{Z}$ as an abelian group;
  - If E is a "sufficiently large" injective module over a left noetherian ring, then it is a saturated model of its complete theory;
  - The abelian groups $\overline{\mathbb{Z}_p} \oplus \mathbb{Q}^{(\kappa)}$, κ infinite.
- If N is a saturated module, then it is pure-injective.
  - because a saturated module is clearly equationally compact!
- In particular, every module can be purely embedded in a pure injective module (any saturated elementary extension).

Two kinds of hulls...

Definition/Theorem

Let M be any module.
There is a unique up-to-isomorphism pure-injective pure extension $N = \overline{M}$ of M such that for any pure-injective pure extension N' of M, there is a pure embedding of N into N' fixing M.
The pure-injective envelope of M.

Definition/Theorem

Let $M \leq N$ be modules, with N pure-injective.
There is a unique (up to isomorphism of N fixing M) minimal pure-injective pure submodule $H(M)$ of N containing M,
the hull of M in N.

Both of these are fundamental tools of the model theory of modules.

Contrasting examples

Consider $\mathbb{Z}_{(p)}$ and $\mathbb{Z}$ as abelian groups.

- $\mathbb{Z}_{(p)}$ is not pure in $\mathbb{Q}$;
  $H(\mathbb{Z}_{(p)}) = \mathbb{Q}$.
- $\mathbb{Z}_{(p)}$ is pure in $\overline{\mathbb{Z}_{(p)}}$, and this is in fact the pure-injective envelope.

- The injective envelope of $\mathbb{Z}$ is $\mathbb{Q}$.
- The pure-injective envelope of $\mathbb{Z}$ is the profinite completion of $\mathbb{Z}$, an abelian group of cardinality the continuum.

The End