Strongly Minimal Steiner Systems: Coordinatization and Quasigroups University of Manitoba

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Overview

1. Quasi-groups and Steiner systems
2. Strongly Minimal Theories
3. Constructing Flat Geometries
4. Coordinatization by varieties of algebras
5. Strongly Minimal (non-associative) Quasigroups

Thanks to Joel Berman, Gianluca Paolini,Omer Mermelstein, and Viktor Verbovskiy.

Quasi-groups and Steiner systems

Definitions

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$.

Connections with number theory

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

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Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939) Skolem ($6n + 1$, 1958)

Similar divisibility conditions for other values of t, k, n.
Keevash 2014: for any t and sufficiently large n, if k is not obviously blocked, there are (t, k, n)-Steiner systems.

Linear Spaces

Definition (1-sorted)

The vocabulary τ contains a single ternary predicate R, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems:

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

$\boldsymbol{K}^*$ includes infinite linear spaces.

Groupoids and semigroups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

Strongly Minimal Theories

STRONGLY MINIMAL

Definition

A complete theory T is strongly minimal if every definable set of every model of T is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Usually this acl pre-geometry is not definable.

The trichotomy

Zilber Conjecture

1. The acl-geometry of every model of a strongly minimal first order theory is
   1. disintegrated (lattice of subspaces distributive)
   2. vector space-like (lattice of subspaces modular)
   3. ‘bi-interpretable’ with an algebraically closed field (non-locally modular)
2. Plus
   1. Flat: Combinatorial class
   2. Are there more? Can this one be refined?

Hrushovski’s showed there are non-locally modular examples which are far from being fields; they don’t admit an infinite associative operation.

Strongly minimal linear spaces

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k.

Question

Are minimal quasigroups Steiner k-systems?

Specific Strongly minimal Steiner Systems

Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(2, k, \infty)$ Steiner-systems.

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Constructing Flat Geometries

The Kueker-Laskowski Formulation

1. Let σ be a finite relational vocabulary. A class $(\boldsymbol{L}_0, \leq)$ of finite structures, with a transitive relation $\leq$ on $\boldsymbol{L}_0 \times \boldsymbol{L}_0$ is called smooth if $B \leq C$ implies $B \subseteq C$ and
   $B \leq C$ is defined by universal sentences:
   for all $B \in \boldsymbol{L}$ there is a collection $p^B(\mathbf{x})$ of universal formulas with $|\mathbf{x}| = |B|$ and for any $C \in \boldsymbol{L}_0$ with $B \subseteq C$,
   $$B \leq C \leftrightarrow C \models \phi(\mathbf{b})$$
   for every $\phi \in P^B$ and $\mathbf{b}$ enumerates B.
2. A structure A is a $(\boldsymbol{L}_0, \leq)$-union if $A = \bigcup_{n<\omega} C_n$ where each $C_n \in \boldsymbol{L}_0$ and $C_n \leq C_{n+1}$ for all $n < \omega$.
3. A structure A is a $(\boldsymbol{L}_0, \leq)$-*generic* if A is a $(\boldsymbol{L}_0, \leq)$-union and for any $B \leq C$ each in $\boldsymbol{L}_0$ and $B \leq A$ there is a $\leq$-embedding of C into A.

Fraïssé, Hrushovski, 'atomic models'

Hrushovski construction for linear spaces

$\mathbf{K}_0^*$ denotes the collection of finite linear spaces in the vocabulary $\tau = \{R\}$.
A line in a linear space is a maximal R-clique
$L(A)$, the lines based in A, is the collections of lines in (M, R) that contain 2 points from A.

Definition: Paolini's δ

[Pao21] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

Amalgamation and Generic model

$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding amalgamation there is unique countable generic.

Theorem: Paolini [Pao21]

There is a generic model for $\boldsymbol{K}_0$; it is ω-stable with Morley rank ω.

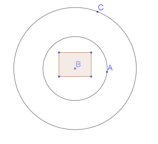
This requires a different notion of 'free amalgamation' than in the Hrushovski construction.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

In Hrushovski's examples the base is unique. But not in linear spaces.
α is the isomorphism type of $(\{a, b\}, \{c\})$,
with $R(a, b, c)$.
An *extended base* for an instance of α is the intersection of the line through $\{a, b\}$ with A.

Overview of construction

Realization of good pairs

1. A good pair C/B *well-placed* by A in a model M, if $B \subseteq A \leq M$ and C is 0-primitive over X.
2. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.

Classes of Structures

I. $\boldsymbol{K}_0^*$: all finite linear τ-spaces.

II. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily ≥ 0.

III. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: $\chi_M(A/B) \leq \mu(A/B)$ μ bounds the number of disjoint realizations of a 'good pair'.

IV. $\boldsymbol{K}_\mu = \mathrm{mod}(T_\mu)$ strongly minimal.

If C/B is well-placed by $A \leq M$, $\chi_M(C/B) = \mu(C/B)$

Basic case

$\boldsymbol{\alpha}$ is the isomorphism class of the good pair $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Context

Let $\mathcal{U}$ be a collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:

1. (i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;
2. (ii) a number $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$

T_μ is the theory of a strongly minimal Steiner $(\mu(\boldsymbol{\alpha}) + 2)$-system

If $\mu(\boldsymbol{\alpha}) = 1$, T_μ is the theory of a Steiner triple system bi-interpretable with a Steiner quasigroup.

$\boldsymbol{K}_\mu$

Definition

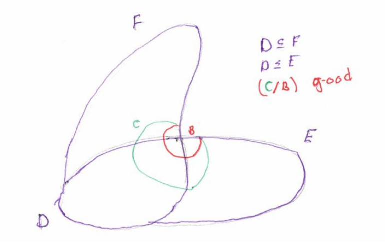
For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).

Theorem (Baldwin-Paolini)[BP20]

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory T_μ.
If $\mu(\boldsymbol{\alpha}) = k$, all lines in any model of T_μ have cardinality $k + 2$.
Thus each model of T_μ is a Steiner k-system and $\mu(\boldsymbol{\alpha})$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. $\boldsymbol{\alpha}$).

The Amalgamation



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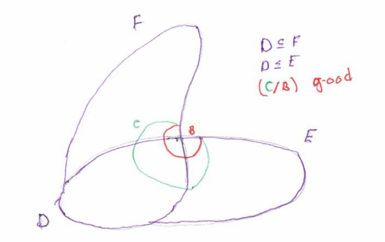

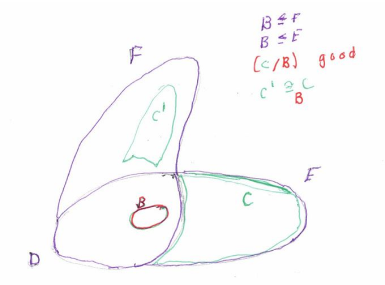

Coordinatization by varieties of algebras

Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras V '(weakly) coordinatizes' a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatized

A collection of algebras V definably coordinatizes a class $\mathcal{S}$ of k-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

Coordinatizing Steiner triple systems

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A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

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Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct a, b, c:

$$R(a, b, c) \text{ if and only if } a * b = c$$

Theorem

Every strongly minimal Steiner (2,3)-system given by T_μ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroup definable in the system.

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on F by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Definition [Pad72]

An (r, k) variety is one in which every r-generated algebra has cardinality k and is free generated by every n-elements.

Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality k, each $A \in V$ determines a Steiner k-system.
(The 2-generated subalgebras.)
And each Steiner k-system admits a weak coordinatization.

Can this coordinatization be definable in the strongly minimal (M, R)?

Forcing a prime power

Fact (Ganter-Werner et al)

1. [Ś61, Grä63]The only (r, k) varieties are those where $r = 0$, $k = 0$; $r = k$; $r = 2$, $k = q = p^n$, for a prime p and a natural number n; $r = 3$, $k = 4$.
2. [GW75, GW80] For each q, the class of q-Steiner systems is coordinatized by a $(2, q)$-variety of block algebras

Proof: As, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

Definability

Definability Theorem

Suppose q is a prime power and $\mu(\alpha) = q - 2$. Then

1. Each $(M, R) \models T_\mu$ is *coordinatizable* by an algebra $(Q_M, *)$ in V.
2. $R(x, y, z)$ is definable in $(Q_M, *)$ by the formula $\theta_F(x, y, z)$ that is the disjunction of the terms $z = f_i(x, y)$ where the $f_i(x, y)$ list the terms generating $F = F_2(V)$. Thus, (M, R) is definable in $(Q_M, *)$.
3. There is an (incomplete) first order theory $\check{T}_\mu$ in the vocabulary $\{*\}$ such that each model of T_μ is coordinatized by a model of $\check{T}_\mu$.

Proof

1. 1) and 2)are immediate from the general coordinatization theorem.
2. 3) Let $\Delta_F(x, y, f_1(x, y), \ldots f_k(x, y))$ be the quantifier-free diagram of F. By 2-transitivity of F_2 , any x, y does. Axiomatize $\check{T}_\mu$ by:

$$Eq(V) \cup \{(\forall x, y)\Delta_F(x, y, f_1(x, y), \ldots f_k(x, y))\} \cup \{\phi \restriction (R/\theta_F) : \phi \in T_\mu\}$$

Non-definability

Theorem: (B) Non-definability in (M, R)

If $\mu(\alpha) = k > 1$ this coordinatization is not definable in (M, R).

Proof

Without loss of generality, let (M, R) be the countable generic and suppose it is coordinatized by $(Q_M, *)$.
Let $\{a, b\}$ be a strong substructure of (M, R) (i.e. $d(\{a, b\} = 2)$ and let $c_1, \ldots c_k$ fill out the line through a, b to a structure A. By genericity there is a strong embedding of A into M.

Then all triples a, b, c_i realize the same quantifier free R-type and $A \leq M$ implies for any permutation ν of k fixing $0, 1$, for $2 \leq i < k$, there is an automorphism of (M, R) fixing a, b and taking c_i to $c_{\nu(i)}$.
Thus, $a * b$ cannot be definable in (M, R).

No definable binary function/elimination of imaginaries

Theorem (B-Verbovskiy)

Suppose T_μ has only a ternary predicate (3-hypergraph) R. If T_μ is either in

1. Hrushovski's original family of examples
2. or one of the B-Paolini Steiner systems

and also satisfies:

1. $\mu \in \mathcal{U}$
2. If $\delta(B) = 2$, then $\mu(B/C) \geq 3$ except
3. $\mu(\boldsymbol{\alpha}) \geq 2$ (for linear spaces)

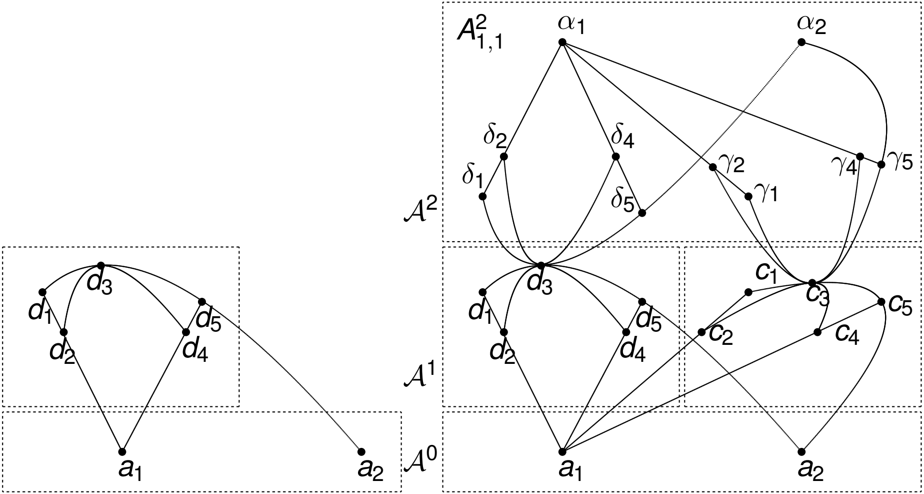
(I) There is no binary function $\emptyset$-definable in T_μ.

(II) In the Hrushovski case, if condition 2) is dropped, there is still no commutative binary function $\emptyset$-definable in T_μ.

Example: definable binary function

Definable binary functions may appear when $\mu(A/B) = 2$ and $d(B) = 2$ is allowed. We put the following lines: $\{a_1, d_2, d_1\}$, $\{a_1, d_4, d_5\}$, $\{a_2, d_5, d_3, d_1\}$, and $\{d_2, d_3, d_4\}$. The elements c_i is the isomorphic copy of d_i over $\{a_1, a_2\}$, for each i. In order to construct $A^2_{1,1}$ we put that α_i is the copy of a_i, δ_i is the copy of d_i as well as γ_i is the copy of c_i for each appropriate i, where the isomorphism under consideration is over $\{d_3, c_3\}$. Then $\alpha_1 \in \mathrm{dcl}^*(a_1, a_2)$.

Diagram: definable binary function



Questions

The choice of the block algebra variety in 1) is not unique. Ganter and Werner [GW80, page 7] describe two different varieties of block algebras (one commutative and one not) over F_5, depending on the choice of the primitive element a of F_5. Thus $\check{T}_\mu$ is not complete. Our constructions show there are continuum many first order theories of strongly minimal block algebras.

Questions

1. Are all the $(Q_M, *)$ (for the same F) elementarily equivalent? in the same (equationally complete?) variety?
2. Do they represent continuum many distinct varieties? I.e, are the classes $HSP(\mathcal{G}_\mu)$ distinct for (sufficiently) distinct μ?

Other model theory of Quasigroups

Barbina-Casanovas building on Cameron

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are the graphs of partial Steiner quasigroups.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T^*_{sq}.
2. T_{sqg} has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and locally finite;
   4. T_{sqg} has TP_2 and $NSOP_1$.

Cameron's example is explicitly two-transitive.

Questions on other cases

1. Is it possible to develop a theory of q-block algebras for arbitrary prime powers similar to that for Steiner quasigroups in [BC1x]? That is, to find a model completion for each of the various varieties of block algebras?
2. Are there strongly minimal $(3,4)$-Steiner systems? Is there a definable coordinatization?

Strongly minimal block algebras $(M, R, *)$

Theorem: Baldwin

For every prime power q there is a strongly minimal Steiner q-system (M, R) whose theory is interpretable in a strongly minimal block algebra $(M, R, *)$.

Let τ' contain ternary relations R, H.

Definition

Fix a prime power q and a $(2, q)$-variety V of quasigroups. Let F_2 denote the free algebra in V on 2 generators. Let $\boldsymbol{K}^{\tau'}_{\mu,V}$ be the collection of finite (R, H)-structures A such that

1. (A, R) is a linear space, such that each non-trivial line has q points.
2. If $A{\restriction}R$ is a maximal clique (line) with respect to R (necessarily $|A| = q$), $A{\restriction}H$ is the graph of F_2.

Construction

Define for each $A' \in \boldsymbol{K}^{\tau'}_{\mu,V}$, let

$$A = A' \restriction R$$

and

$$\delta_{\tau'}(A') = \delta_{\tau}(A)$$

and induce $\leq'$ from δ'. Note that B/A is a good pair, just when B'/A' is a good pair. Since both the restriction $\delta(A) \geq 0$ and the bound imposed by μ are universally axiomatized it is easy to check that $(\boldsymbol{K}^{\tau'}_{\mu,V}, \leq')$ is smooth. However it is AE-axiomatized because of clause 2. The amalgamation now goes as in the base case.

Universal Algebra

Question

$T_{\mu',V} \restriction \tau$ is not T_μ.
Is it possible to characterize those μ such that T_μ can be interpreted in a quasigroup?

Different varieties of quasigroups may have the same free algebra on two generators. The construction depends on both the original $\boldsymbol{K}_\mu^q$ and $F_2(V)$.

How many varieties can arise from the same $F_2(V)$? There are two variants on this question.
One is, 'how many varieties of quasigroup can have the same free algebra on two generators?'. The second asks only of varieties $V(Q, H)$ that arise from a theory T_μ as constructed here.

Universal Algebra II

Corollary

For each $T_{\mu',V}$ with prime power line length, any $M \models T_\mu$, the reduct to $*$ is in a variety (that is congruence permutable, regular and uniform [Qua76, Theorem 3.1] or [GW75, Corollary 2.4] but not residually small [BM88, Corollary 8].

Question

Every finite algebra in a $(2, q)$ has a finite decomposition into directly irreducible algebras ([GW75, Corollary 2.4]. *Are there any similar results for infinite strongly minimal block algebras?*

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