STRONGLY MINIMAL STEINER SYSTEMS II: COORDINATIZATION AND QUASIGROUPS

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Abstract. We note that a strongly minimal Steiner k-Steiner system (M, R) from [BP20] can be 'coordinatized' in the sense of [GW75] by a quasigroup if k is a prime-power. But for the basic construction this coordinatization is never definable in (M, R). Nevertheless, by refining the construction, if k is a prime power there is a $(2, k)$-variety of quasigroups which is strongly minimal and definably coordinatizes a Steiner k-system.

A *linear space* is collection of points and lines that satisfy a minimal condition to call a structure a geometry: two points determine a line. A linear space is a Steiner k-system if every line (block) has cardinality k. Such mathematicians as Steiner, Bose, Skolem, Bruck have established deep connections between the existence of a Steiner system with v points and blocks of size k and divisibility relations among k and v.

A strongly minimal theory is a complete first order theory such that every definable set is finite or cofinite. They are the building blocks of $\aleph_1$-categorical theories. Alternatively, in a strongly minimal theory T the model theoretic notion of algebraic closure[1] determines a combinatorial geometry (matroid). Zilber conjectured that these geometries were all discrete $(\mathrm{acl(A)} = \bigcup_{\mathrm{a}\in\mathrm{A}} \phi(\mathrm{x},\mathrm{a}))$, locally modular (group-like) or field-like. The examples here are based on Hrushovki's construction refuting this conjecture [Hru93]. These counterexamples, with 'flat[2] geometries' [Hru93, Section 4.2], have generally been regarded as an incohate class of exotic structures. Indeed, a distinguishing characteristic is the inability to define an associative operation.

It is easy to see that for any strongly minimal linear space there is a k such all lines have length at most k. With Paolini [BP20], we varied the Hrushovski construction to find strongly minimal k-Steiner systems for all $k \geqslant 3$, each line has exactly k points.

We showed in Section 2 of [BP20][3] that linear spaces can be naturally formulated in a one-sorted logic with single ternary 'collinearity' predicate and we proved:

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[1] $a \in \mathrm{acl}(B)$ if for some $\phi(x, \mathbf{b})$ with $\mathbf{b} \in B$, $\phi(a, \mathbf{b})$ and $(\exists^{<} kx\phi(x, \mathbf{b})$ for some k.

[2] The dimension of a closed subspace is determined from its own closed subspaces by the inclusion-exclusion principle.[BP20, Definition 3.8].

[3] We use this paper as a common reference for many earlier results and definitions that are scattered in the literature.

Fact 0.1 ([BP20]). *For each k, with $3 \leqslant k < \omega$, there are $2^{\aleph_0}$ strongly minimal theories T_μ (depending[4] on an integer valued function μ) of infinite linear spaces in the one-sorted vocabulary τ whose models are Steiner k-systems.*

These theories are model complete and satisfy the usual properties of counterexamples to Zilber's trichotomy conjecture. Their acl*-geometries are non-trivial, not locally modular, and flat.*

Much of the history of Steiner systems interacts with the general study of non-associative algebraic systems such as quasigroups. A quasigroup is a structure with a single binary operation whose multiplication table is a Latin Square (each row or column is a permutation of the universe). Stein [Ste56] made deep connections between Steiner k-systems and quasigroups. A line of work from the 1950-1980's including [Ste56, Grä63, GW75, Eva82] established that Steiner k-systems were 'informally coordinatized' by varieties of quasigroups if and only if k is a prime power. We extend these results to infinite quasisgroups and Steiner system of every cardinality with two contrasting results.

Theorem 0.2 ([BV20]). *[Theorem 3.11] Let T_μ be a strongly minimal theory of Steiner k-systems (M, R).*

(1) If k is a prime power, M, R is definable in a model of a first order theory $\check{T}^\mu$.
(2) Unless $k = 3$, the 'coordinatizing quasigroup' is not definable in (M, R).

Nevertheless when $k = q$ is a prime power we can find strongly minimal Steiner systems that are definable in strongly minimal quasigroups.

Theorem 0.3. *(Theorem 4.3) For each q and each of the T_μ in Theorem 0.1 with line length $k = q = p^n$ and certain varieties of quasigroups V, there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that defines a strongly minimal q-Steiner system.*

This result rests primarily on work of [GW75, Ste56, Ś61] and others who achieved a 'coordinatization' of such Steiner systems by quasigroups. The contribution here is that although, for $k > 3$, this coordinatization is not invertible, the Steiner system never defines a quasigroup, we can in fact demand for $k = q = p^n$ the existence of a Steiner k-system that is defined in a strongly minimal quasigroup. The key to this is the relationship of so-called $(2, k)$ varieties [Pad72, GW75, Qua92] to a two-transitive finite structure and thus eventually to the reconstruction of a finite (near)-field.

We discussed in the introduction and Remark 5.27 of [BP20] the connections of this work with [BC1x, CK16, HP]. These works construct first order theories of Steiner systems or projective planes that are at the other end of the stability spectrum from those here. Evans [Eva04] uses the Hrushovski construction to address combinatorial issues about Steiner systems.

This paper depends heavily on the results and notation of [BP20, BV20]. Certain arguments will require consulting those papers. We acknowledge helpful discussions with Joel Berman, Omer Mermelstein, Gianluca Paolini, and Viktor Verbovskiy.

[4] The theory of course depends on the line length k; but it is coded by μ so we suppress the k.

1. Coordinatization

Descartes's coordinatizaton identified addition and multiplication of line segments in a Euclidean plane as algebraic operations on what became over the next 300 years a (sub)field (which one depends on the geometric axioms) of the real numbers. Hilbert, building on Von Staudt, and before the notions were formally defined, established a *bi-interpretation* between an ordered field F where every positive number has a square root and the Euclidean plane coordinatized by F.

Makowsky [Mak19] pointed out the subtleties of the coordinatization notion as a property connecting theories. In particular he emphasized the necessity of establishing the *definability* of the interpretation in each direction verifying the composition is the identity. In the case at hand, there may be an informal coordinatization, but the Decartes direction is *never definable* in the geometric language (for line length $\geqslant 4$).

We contrast two notions.

Definition 1.1. *(1)* [GW75, GW80] *A class of structures (specifically geometries (M, R)) is* coordinatizable *if there is* 1-1 *correspondence between it and a well-behaved class of algebras (specifically quasigroups $(M, *)$).*

(2) The coordinatization is definable *if there is first order formula in R that defines $*$.*

Ganter and Werner identify those classes of finite Steiner systems which are coordinatizable as in Definition 1.1 by certain varieties of quasigroups. But, as they are aware this identification is not unique; the same Steiner-system can be coordinatized using the same method by different algebras (that are not even in the same variety). Thus the theory of the Steiner system does not even predict the equational theory of coordinatizing algebra and certainly does not control the first order theory.

2. Background

We first give a quick survey of a set of notions from combinatorics and universal algebra and then a short introduction to the Hrushovski construction. Then we describe our notation.

A Steiner (t, k, v)-system is a pair (P, B) such that $|P| = v$, B is a collection of k element subsets of P and every t element subset of P is contained in exactly one block. Since we are primarily interested in interested infinite structures, we omit the v unless it is crucial and so, by Steiner k-system I mean Steiner $(2, k)$ system of arbitrary cardinality. A *groupoid*[5] (also called *magma*) is a structure $(A, *)$ with one binary function $*$.

A variety is a collection of algebras (structures in a vocabulary with only function/constant symbols and no relation symbols) that is defined by a family of equations. The essential characteristic of the equational theories below is that each defining equation involves only two variables. In particular, none of the varieties are associative.

[5]In category theory the term groupoid means all morphisms are invertible. Thus, it is more analogous to our 'quasigroup'. But most of the references for this paper use groupoid with no explanation to mean binary function.

Most of the current section[6] is a series of definitions needed to apply the Hrushovski construction as modified in [BP20]. The basic ideas of the Hrushovski construction are i) to modify the Fraïssé approach by replacing substructure by a notion of strong substructure, defined using a predimension δ (Definition 2.5) so that independence with respect to the dimension induced by δ is a combinatorial geometry[7] and ii) to employ a function μ to bound the number 0-primitive extensions of each finite structure so that closure in this geometry is algebraic closure.

We phrase our work in the generalization of the Fraïssé and Hrushovski constructions laid out in [KL92].

Notation 2.1. *(1) A* smooth class $(L_0, \leqslant)$ *is a countable collection of finite structures with a transitive relation on* $\boldsymbol{L}_0$*,* strong substructure *($\leqslant$), refining substructure such that $B \leqslant C$ implies $B \leqslant C'$ if $B \subseteq C' \subseteq C$. However, L_0 need not be closed under substructure. If a smooth class satisfies amalgamation and joint embedding there is a countable generic model M, i.e. if finite A, B are each strong in M, they are automorphic in M.*

(2) Given a class of finite structures L_0, $\hat{L}_0$ denotes the collection of structures of direct limits of members of L_0.

The next notation outlines the dependencies among the variations on the construction that appear below.

Notation 2.2. A *Hrushovski sm-class* is determined by a quintuple $(\sigma, \boldsymbol{L}_{-1}, \epsilon, L_0, \mathbf{U})$. $\boldsymbol{L}_{-1}$ is a collection of finite structures in a vocabulary σ, not necessarily closed under substructure. ϵ is a function from members of $\boldsymbol{L}_{-1}$ to natural numbers satisfying the conditions imposed on δ in Definition 2.5. $\boldsymbol{L}_0$ is a subset of $\boldsymbol{L}_{-1}$ defined using ϵ. From such an ϵ, one defines notions of $\leqslant$, primitive extension, and good pair. Hrushovski gave one technical condition on the function μ counting the number of realizations of a good pair that ensured the theory is strongly minimal rather than ω-stable of rank ω. Fixing a class $\mathbf{U}$ of functions μ satisfying that condition in the base case and others for special purposes provides a way to index a rich group of distinct constructions. At various times $\mathbf{U}$ is instantiated as $\mathcal{U}, \mathcal{B}, \mathcal{C}, \mathcal{F}$ or $\mathcal{T}$. Thus one obtains a strongly minimal theory T_μ and a generic structure $\mathcal{G}_\mu$.

In the specific developments in the paper L becomes K, ϵ becomes δ and a various script letters are substituted for U to describe the counting function μ.

Notation 2.3. *We work in a vocabulary τ with one ternary relation R, and assume always that R can hold only of three distinct elements and then in any order (i.e., a 3-hypergraph). We say a* maximal *R-clique in a structure M is a line.*

We use the words 'block' and 'line' interchangeably and often fail to distinguish when the line has full length. A line is a maximal clique and we may write clique to denote a subset of a line.

Definition 2.4. *(1) $\boldsymbol{K}_{-1}$ is the collection of* linear spaces*, τ-structures such that 2-points determine a unique line (maximal clique); we interpret R as collinearity. By convention two unrelated elements constitute a* trivial line*.*

[6]The two page Section 2.1 of [BP20] summarises the role of strongly minimal sets in model theory and how strongly minimal Steiner systems arise.

[7]The requirement that the range of this function is well-ordered is essential to get the exchange property in the geometry; using rational or real coefficients yields a stable theory and the dependence relation of forking [BS96].

(2) *For* $\ell \subseteq A$*, we denote the cardinality of a clique* ℓ *by* $|\ell|$*, and, for* $B \subseteq A$*, we denote by* $|\ell|_B$ *the cardinality of* $\ell \cap B$*.*
(3) *We say that a non-trivial line* ℓ *contained in* A *is* based in $B \subseteq A$ *if* $|\ell \cap B| \geqslant 2$*, in this case we write* $\ell \in L(B)$*.*
(4) *The* nullity of a line ℓ *contained in a structure* $A \in \boldsymbol{K}_{-1}$ *is:*

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

Now we define our geometrically based pre-dimension function [Pao21].

Definition 2.5. *(1) Recall* $\boldsymbol{K}_{-1}$ *is the collection of finite linear spaces. For* $A \in \boldsymbol{K}_{-1}$ *let:*

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

(2) *Let:*

$$\boldsymbol{K}_0 = \{A \in \boldsymbol{K}_{-1} \text{ such that for any } A' \subseteq A, \delta(A') \geqslant 0\},$$

and $(\boldsymbol{K}_0, \leqslant)$ *be as in* [BS96, Definition 3.11]*, i.e. we let* $A \leqslant B$ *if and only if:*

$$A \subseteq B \wedge \forall X(A \subseteq X \subseteq B \Rightarrow \delta(X) \geqslant \delta(A)).$$

(3) *We write* $A < B$ *to mean that* $A \leqslant B$ *and* A *is a proper subset of* B*.*
(4) *We say that* B *is a* primitive extension *of* A *if* $A \leqslant B$ *and there is no* B_0 *with* $A \subsetneq B_0 \subsetneq B$ *such that* $A \leqslant B_0 \leqslant B$*.*

The remainder of this section is not needed until Section 4. The class $\boldsymbol{K}_0$ satisfies amalgamation with the following construction.

Definition 2.6. [BP20, Lemma 3.14] *Let* $A \cap B = C$ *with* $A, B, C \in \boldsymbol{K}_0$*. We define* $D := A \oplus_C B$ *as follows:*

(1) *the domain of* D *is* $A \cup B$*;*
(2) *a pair of points* $a \in A - C$ *and* $b \in B - C$ *are on a non-trivial line* ℓ' *in* D *if and only if there is line* ℓ *based in* C *such that* $a \in \ell$ *(in* A*) and* $b \in \ell$ *(in* B*). Thus* $\ell' = \ell$ *(in* D*).*

The following definition describes the pairs $B \subseteq C$ such that eventually B/C will be an algebraic set (realized only finitely often).

Definition 2.7. *Let* $A, B \in \boldsymbol{K}_0$ *with* $A \cap B = \emptyset$ *and* $A \neq \emptyset$*.*

(1) *When we have a pair* A, C *with* $A \leqslant C$ *we often write* $\hat{C}$ *for* $C - A$ *to simplify notation.*
(2) B *is a* primitive extension *of* A *if* $A \leqslant B$ *and there is no* $A \subsetneq B_0 \subsetneq B$ *such that* $A \leqslant B_0 \leqslant B$*.*
  B *is a* k-primitive extension *if, in addition,* $\delta(B/A) = k$*.*
  We stress that in this definition, while B *may be empty,* A *cannot be.*
(3) *We say that the* 0*-primitive pair* A/B *is* good *if there is no* $B' \subsetneq B$ *such that* (A/B') *is* 0*-primitive. (This notion was originally called a minimal simply algebraic or m.s.a. extension.)*
(4) *If* A *is* 0*-primitive over* B *and* $B' \subseteq B$ *is such that we have that* A/B' *is good, then we say that* B' *is a* base *for* A *(or sometimes for* AB*).*
(5) *If the pair* A/B *is good, then we also write* (B, A) *is a* good pair.

The following notation singles out the effect of the fact that our rank depends on line length rather than the number of occurrences of a relation.

We single out a type of good pair that provides the invariant for the Steiner systems. As noted in [BP20], T_μ is a Steiner k-system if $\mu(\boldsymbol{\alpha}) = k - 2$.

Notation 2.8 (Line length). *We write $\boldsymbol{\alpha}$ for the isomorphism type the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$.*

By Lemma 5.18 of [BP20]*, lines in models of T_μ have length k if and only if $\mu(\boldsymbol{\alpha}) = k - 2$.*

Definition 2.9. *Good pairs were defined in Definition 2.7.*

(1) Let $\mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:
  - *(i) a number $\mu(\beta) = \mu(B, C) \geqslant \delta(B)$, if $|C - B| \geqslant 2$;*
  - *(ii) a number $\mu(\beta) \geqslant 1$, if $\beta = \boldsymbol{\alpha}$.*

(2) For any good pair (B, C) with $B \subseteq M$ and $M \in \hat{\boldsymbol{K}}_0$, $\chi_M(B, C)$ denotes the number of disjoint copies of C over B in M. Of course, $\chi_M(B, C)$ may be 0.

(3) Let $\boldsymbol{K}_\mu$ be the class of structures M in $\boldsymbol{K}_0$ such that if (B, C) is a good pair $\chi_M(B, C) \leqslant \mu((B, C))$.

(4) $\hat{\boldsymbol{K}}_\mu$ is the class of direct limits of structures in $\boldsymbol{K}_\mu$.

Fact 2.10. [BP20] *For each $\mu \in \mathcal{U}$, the class $\boldsymbol{K}_\mu$ has amalgamation (and joint embedding) so there is a countable generic model $\mathcal{G}_\mu$ whose theory T_μ is a strongly minimal Steiner system where $k = \mu(\boldsymbol{\alpha}) + 2$.*

3. Associating Strongly Minimal Steiner systems with Quasigroups

Following [GW75, GW80], we say a class of structures is *coordinatizable* if there is 1-1 correspondence between it and a well-behaved class of algebras (3.9). We explore around Fact 3.16 the connections between this notion and the concept of an *interpretation* [Hod93].

Definition 3.1 ([Smi07]). *A quasigroup*[8] *$(Q, *)$ is a groupoid*[9] *$(A, *)$ such that for $a, b \in Q$, there exist unique elements $x, y \in Q$ such that both*

$$ax = b, ya = b.$$

the general notion is a universal Horn class, not a variety. See Definition 3.6 and Remark 3.7.

We will discuss in detail three (families of) varieties (equational classes) of quasigroups.

Definition 3.2. [Smi07]

1. *A Steiner quasigroup a groupoid which satisfies the equations: $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.*
2. *A Stein quasigroup*[10] *is a combinatorial quasigroup with the additional equations: $x * x = x$, $(x * y) * y = y * x$, $(y * x) * y = x$.*

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[8] Alias: multiplicative quasigroup [MMT87], combinatorial quasigroup [Smi07].

[9] In the background literature on quasigroups, a *groupoid* is simply a set with a binary operation. So, I use this notation although it is no longer common.

[10] [RR10] points out that the term Stein quasigroup is used in two ways and refers to the usage here as an S^*-quasigroup.

(3) *Given a (near)-field*[11] $(F, +, \cdot, -, 0, 1)$ *of cardinality* q *and a primitive element* $a \in F$*, define a multiplication* $*$ *on* F *by* $x * y = y + (x - y)a$*. An algebra* $(A, *)$ *satisfying the 2-variable identities of* $(F, *)$ *is a* block algebra [GW80] *over* $(F, *)$*. Note* $(F, *)$ *is idempotent.*

While every group is an quasigroup, the Stein and Steiner quasigroup are rather special quasigroups since they are idempotent. Thus, a Stein or Steiner quasigroup $(Q, *)$ cannot be a group unless it has only one element.

Steiner triple systems and Steiner quasigroups are actually *interdefinable.*

Fact 3.3. *Each Steiner* triple *system is interdefinable with a Steiner quasigroup (Definition 3.2).*

Proof. Given the algebra, the blocks are the 2-generated subalgebras; given a Steiner triple system, let $x \circ y$ be the third element of the block if $x \neq y$ and $x \circ x = x$. Since all blocks are isomorphic to the unique 3 element Steiner quasigroup, the resulting algebra is a Steiner quasigroup.

■

Corollary 3.4. *There are* $2^{\aleph_0}$ *strongly minimal theories* T_μ *of Steiner quasigroups and so non-isomorphic Steiner triple systems of cardinality* $\aleph_0$*.*

Proof. We have an explicit 1-dimensional (the domain and range of the interpretation is the universe) bi-interpretation between Steiner triple systems and the Steiner 3-systems that were given by Theorem 0.1. ■

While these algebras are in the variety of Steiner triple system, for each μ we have selected a single algebra in each uncountable cardinality. So we are distinguishing first order theories not varieties. The following example illustrates the issue addressed in more generality and more detail in Fact 3.11.

Fact 3.5 (Stein quasigroups)**.** [GW80, page 5] *Each* $(2, 4)$*-Steiner system is coordinatized by a Stein quasigroup,*

Proof sketch: One direction is obvious; the blocks are the 2-generated subalgebras of the quasigroup. For the other direction, the universe of the algebra is $Q = P$. For each block $b \in B$, fix a four element Stein quasigroup on B. We consider two possibilities: A_1 requires $0 * 1 = 3$ while A_1 requires $0 * 1 = 2$. Regardless of the choice of A_i, the entire structure is a Stein quasigroup. It clearly satisfies the three equations of Definition 3.2.2 because they involve elements only within a single block and also the requirement that each equation $ax = b$ $(ya = b)$ has a unique solution, as again the solution is within the block determined by a, b.

Following [GW75, Pad72], we say

Definition 3.6. (1) *The variety* V *is an* (r, k)*-variety if every* r*-generated subalgebra of any* $A \in V$ *is isomorphic to the free* V*-algebra on* r*,* $F_r(V)$ *and* $|F_r(V)| = k$*.*

(2) *A* Mikado *variety* [GW75, 128] *is* $(2, q)$*-variety with all fundamental operations binary and with an equational base of* 2*-variable equations.*

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[11] A near-field is an algebraic structure satisfying the axioms for a division ring, except that it has only one of the two distributive laws. They were introduced by Dickson in 1905; we focus on the field case. We have seen the term 'block algebra' only in [GW80], but it seems appropriate. I see no relation with the notion of 'block algebra' arising from the blocks of a group representation.

This is one of 5 equivalent characterizations of an (r,k) variety in [Pad72]. Obviously, the collection of r-generated subalgebras $A \in V$ form an Steiner (r,k)-system; we need a third: the automorphism group of any r generated algebra is strictly (i.e. sharply) r-transitive.

Remark 3.7. In general a sub-quasigroup of an quasigroup $(Q,*)$ need not be a quasigroup. But if $V(Q)$ (i.e. HSP(Q)) is an (r,k) variety, then every algebra in $V(Q)$ is a quasigroup [Qua92, Theorem 3]. So in this paper[12] we can regard quasigroups as structures with one binary operation.

We rely heavily on a 'classical' observation of Trevor Evans. It requires no proof, but a little thought. Evans [Eva82] calls a variety V *binary* if both all function symbols of V are binary and the defining equations involve only 2 variables.

Fact 3.8 ([Ste64, Eva76])**.** *If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.*

In this context, Definition 1.1 becomes:

Definition 3.9. *A variety (equational class) or more generally a first order theory of algebras V* coordinatizes *a class $\mathcal{S}$ of $(2,k)$-Steiner systems if:*

The universe M of each member (M,R) of $\mathcal{S}$ is the domain of an algebra $(M,*)$ in V and the lines are the 2-generated $*$-subalgebras.

If $*$ is definable from R, this is a definable coordinatization*.*

We sketch the proof the following old results to clarify the situation.

Fact 3.10.
1. [Ś61] *The only (r,k) varieties are those where $r=0$, $k=0$; $r=k$; $r=2$, $k=q=p^n$, for a prime p and a natural number n; $r=3$, $k=4$.*
2. [GW75, GW80] *For each q, the class of q-Steiner systems is coordinatized by a $(2,q)$-variety of block algebras (Definition 3.2).*

Proof sketch: Only Steiner $(2,q)$-systems with $q=p^n$ for some prime p, and $n \geqslant 1$ are relevant here. It is easy to check that the block algebras defined in Definition 3.2 are $(2,k)$ algebras. But, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out, an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Given the Steiner q-system we assign to each line a copy of the unique q element algebra $F_2(V)$. This gives an algebra in V by Fact 3.8 $\blacksquare_{3.10}$

We easily see 1) of Theorem 3.11 from Facts 3.8 and 3.10.

Theorem 3.11. *If T_μ is a strongly minimal Steiner k-system (from Fact 0.1) and V is a Mikado $(2,k)$ variety of quasigroups, then*

1. *Each $(M,R) \models T_\mu$ is* coordinatizable *by an algebra $(Q_M,*)$ in V.*
2. *$R(x,y,z)$ is definable in $(Q_M,*)$ by the formula $\theta_F(x,y,z)$ that is the disjunction of the terms $z=f_i(x,y)$ where the $f_i(x,y)$ list the terms generating $F=F_2(V)$. Thus, (M,R) is definable in $(Q_M,*)$.*

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[12] In general the variety generated by a quasigroup contains groupoids that are not quasigroups. See [Qua92], [MMT87, page 126], [SR99, Example 2.2]. In the general situation, the requirement that the binary relation has inverses must be enforced by binary left and ring division operators.

(3) *There is an (incomplete) first order theory* $\check{T}_\mu$ *in the vocabulary* $\{*\}$ *such that each model of* T_μ *is coordinatized by a model of* $\check{T}_\mu$.
(4) *If* $\mu(\boldsymbol{\alpha}) = k > 1$ *this coordinatization is not definable in* (M, R).

Proof. 1) is immediate from Fact 3.10. The statement of 2) defines the interpretation.

3) Let $\Delta_F(x, y, f_1(x, y), \dots f_k(x, y))$ denote the quantifier-free diagram of F. The two-transivity of F guarantees the particular choice of the two elements x, y does not matter. $\check{T}_\mu$ is axiomatized by $Eq(V) \cup \{(\forall x, y)\Delta_F(x, y, f_1(x, y), \dots f_k(x, y))\} \cup \{\phi\restriction(R/\theta_F) : \phi \in T_\mu\}$.

4) Without loss of generality, let (M, R) be the countable generic and suppose it is coordinatized by $(Q_M, *)$. Let $\{a, b\}$ be a strong substructure of (M, R) (i.e. $d(\{a, b\} = 2)$ and let $c_1, \dots c_k$ fill out the line through a, b to a structure A. By genericity there is a strong embedding of A into M.

Then all triples a, b, c_i realize the same quantifier free R-type and $A \leqslant M$ implies for any permutation ν of k fixing $0, 1$, for $2 \leqslant i < k$, there is an automorphism of (M, R) fixing a, b and taking c_i to $c_{\nu(i)}$. Thus, $a*b$ cannot be definable in (M, R). ∎

Despite Theorem 3.11.4, which shows there is no reason to think $\mathrm{Th}(\mathrm{Q_M}, *)$ is strongly minimal, we find strongly minimal theories of quasigroups in Section 4.

QUESTION 3.12. We have found a coordinatizing algebra Q_M for each model M of T_μ. The construction depends on M and a particular free algebra F on two generators. The choice of the block algebra variety in 1) is not unique. Ganter and Werner [GW80, page 7] describe two different varieties of block algebras (one commutative and one not) over F_5, depending on the choice of the primitive element a of F_5 (Definition 3.2. Thus $\check{T}_\mu$ is not complete. Our constructions (Theorem 4.3) show there are continuum many first order theories of strongly minimal block algebras.

(1) *Are all the* $(Q_M, *)$ *(for the same* F*) elementarily equivalent? in the same (equationally complete?) variety?*
(2) *Do they represent continuum many distinct varieties? I.e, are the classes* $HSP(\mathcal{G}_\mu)$ *distinct for (sufficiently) distinct* μ*?*

QUESTION 3.13. The use of the graph of the quasigroup in Construction 4.2 is similar to that in the study of model complete Steiner triple system of Barbina and Casanovas [BC1x]. As noted in Remark 5.27 of [BP20], their generic structure M differs radically from ours: $\mathrm{acl_M(X)} = \mathrm{dcl_m(X)}$.

Is it possible to develop a theory of q-block algebras for arbitrary prime powers similar to that for Steiner quasigroups in their paper? That is, to find a model completion for each of the various varieties of block algebras discussed in Definition 3.2.3?

Since $\check{T}_\mu$ is not complete, so it can't be interpreted in the complete theory T_μ. But there is a much stronger reason for the failure to define $*$ in (M, R). But we need some further hypotheses on μ.

Definition 3.14. *We say* $f(x, y)$ *is a* non-trivial binary definable function *if* f *is definable by a formula* $\phi(x, y, \mathbf{e})$ *and for every* a *there exist* b, b' *such that* $f(a, b) \neq f(a, b')$ *(and similarly reversing the variables).*

The class $\mathcal{T}$ of admissible μ-functions in [BV20] ensures that there are no non-trivial binary (indeed n-ary) functions.

Definition 3.15. *Let $\boldsymbol{K}_{-1}$ be the class of finite linear spaces as in Definition 2.3. Recall from* [BP20] *that we allowed $\mu(\boldsymbol{\alpha}) = 1$ to accommodate Steiner triple systems. We say that the function μ from good pairs into ω satisfies*

(1) $\mu \in \mathcal{U}$ if the standard Hrushovski condition is met: $\mu(A/B) \geqslant \delta(B)$.
(2) μ triples *($\mu \in \mathcal{T}$) if for $\mu(A/B) \geqslant 3$ unless $\delta(B) = 1$ or B is an independent pair.*

With Verbovskiy we introduced the notion of a decomposition of finite G-normal subsets of Hrushovski strongly minimal sets with respect to automorphism groups to prove:

Fact 3.16. *[*[BV20]*] For any strongly minimal Steiner system (M, R)*

(1) If $\mu \in \mathcal{T}$ (μ-triples) there is no definable non-trivial n-ary function definable in (M, R) and so does not have elimination of imaginaries.
(2) If $\mathcal{C} \cap \mathcal{T}$ then for every finite A, $\mathrm{dcl(A)} = \mathrm{A}$ *but (with minor further restriction[13]) on μ), for every finite A $|\mathrm{acl(A)}| = \aleph_0$.*
(3) If $\mu \in \mathcal{U}$ T_μ does not admit elimination of imaginaries. Tentative April 1, 2021: last steps remain to be checked.

Fact 3.16.3 is proved for the basic Hrushovski construction in [BV20]. The crucial distinction between 1) and 2) in Fact 3.16 is that 2) there may be definable binary functions but they cannot be commutative.

In [Bal20] we investigate various combinatorial problems about the classes of quasivarieties constructed here. In particular, we find strongly minimal Steiner triple systems of every infinite cardinality that are two-transitive, so with uniform cycle graphs [CW12], and further that are ∞-sparse in the sense of [CGGW10].

We now show the versatility of the method of construction by finding Steiner systems which both do and don't admit definable unary functions.

Lemma 3.17. *If $\mu \in \mathcal{U}$ and $\mu(\boldsymbol{\alpha}) \geqslant 2$, i.e. lines have length at least 4, there is a 12-element structure A that is 0-primitive over a singleton a. If $\mu(A) = 1$, then T_μ admits a non-trivial definable unary function.*

Proof. Let $\boldsymbol{\epsilon}$ be the isomorphism type of the pair $(\{a\}, \{b, c\} \cup \{d_i : 1 \leqslant i \leqslant 9\})$ where R holds of a, b, c, $ad_{2i+1}d_{2i+2}$ (for $0 \leqslant i \leqslant 3$) $bd_{2i+2}d_{2i+3}$ (for $0 \leqslant i \leqslant 2$), b, d_8, d_1, and finally each triple from $\{c, d_8, d_9, d_4\}$. There are 12 points, nine 3-point line segments and one with 4 points so A denotes the entire structure $\delta(A) = 12 - (9 + 2) = 1$. By inspection, each proper substructure A' has $\delta(A') \geqslant 2$ so A is 1-primitive over $\{a\}$. But d_9 is the unique point that is in exactly one clique within A. Thus, if $\mu(\epsilon) = 1$, the formula $(\exists x_1, x_2 y_1, \ldots y_8)\Delta(x_0, x_1, x_2 y_1, \ldots y_8, y_9)$ (where Δ is the quantifier free diagram of A) defines d_9 over a in any model of T_μ. ∎

While we have given only one example, one can extend the length of the cycle and get infinitely many examples. Note that the construction in Lemma 3.17 is iterable so the definable closure *may not be locally finite*.

Suppose, however that $\mu(\boldsymbol{\epsilon}) \geqslant 2$. Then if $\{a\} \leqslant M$, by [BP20, Corollary 5.16], there are 2 realizations of A over $\{a\}$ and so $d_9 \in \mathrm{acl}(\{\mathrm{a}\} - \mathrm{dcl}(\{\mathrm{a}\}$. Moreover, if $\{a\} \not\leqslant M$ then $d_9 \in \mathrm{acl}(\emptyset)$.

[13] E.g. If the unique 7 element (Fano) plane is in U then $\mathrm{acl}(\emptyset)$ is infinite.

Recall $\mathcal{U}$ is the set of μ such that for any good pair B/A, $\mu(B/A) \geqslant \delta(B/A)$. While the construction with admissible $\mathcal{U}$ does not imply trivial unary closure, we can obtain triviality by changing the class U of admissible μ.

Definition 3.18. *Define $\mathcal{C}$ by restricting $\mathcal{U}$ by requiring that if $|A| = 1$ and some point in B is determined by A (such as $\boldsymbol{\gamma}$ in Lemma 3.17), then $\mu(B/A) = 1$.*

We used the cycle graphs of [CW12] to prove in [BP20, 4.11] that there are $2^{\aleph_0}$ distinct strongly minimal Steiner systems T_μ; this proof remains valid if μ is restricted to $\mathcal{C}$ or even $\mathcal{C} \cap \mathcal{T}$. Clearly amalgamation can not introduce unary functions so we have:

Proposition 3.19. *If $\mu \in \mathcal{C}$, and M is the generic for $\boldsymbol{K}_\mu$, for any $a \in M$,* $\mathrm{dcl(a)} = \{\mathrm{a}\}$.

4. Constructing strongly minimal quasigroups

We have shown that in general the strongly minimal Steiner k-systems for $k > 3$ do not definition a quasigroup. Nevertheless, definition in the other direction may hold. We will show each for $k = q = p^n$, there are strongly minimal Steiner systems that are definable in a strongly minimal quasigroup. Thus in a round-about-fashion we arrive at strongly minimal block algebras, that are in $(2, q)$ varieties and determine Steiner q-systems.

We will obtain this result by jointly constructing a Steiner system (M, R) and a multiplication $*$, requiring that the $*$-algebra be in a given $(2, q)$-variety V (Definition 3.6) that coordinatizes (M, R).

Definition 4.1. *Fix $\mu \in \mathcal{U}$ with $\mu(\boldsymbol{\alpha}) = q-2$ and a $(2, q)$-variety V of quasigroups.*

(1) Let the class $\boldsymbol{K}^q_\mu$ be the finite τ-structures A such that each maximal clique has q-elements. This is expressed by $\forall\exists$ τ-sentence.

(2) Expand τ to τ' by adding a ternary symbol H. Let $\boldsymbol{K}'_\mu$ $(= \boldsymbol{K}'_{\mu',V})$ be the finite τ'- strucures A' such that $A' \restriction \tau \in \boldsymbol{K}^q_{\mu'}$ and $A' \restriction H$ is the graph of $F_2(V)$ on each line.

(3) Let (A/B) be a good pair for $\boldsymbol{K}^q_\mu$ with isomorphism type $\boldsymbol{\gamma}$. Let $\boldsymbol{\gamma'}$ be the isomorphism type of the image of (A/B) in the τ'-structure constructed in the previous paragraph[14]*.*

(4) Note $\mu'(\boldsymbol{\alpha'})$ must be 1, when the τ-reduct is a line of length k, over a two-point base, since two points determine a line. For any other isomorphism type of a good pair with $\boldsymbol{\gamma} = \boldsymbol{\gamma'} \restriction \tau$, let $\mu'(\boldsymbol{\gamma'}) = \mu(\boldsymbol{\gamma})$.

For each prime power $q \geqslant 4$, we show that the class of finite linear spaces such that each maximal non-trivial (at least three elements) clique has size q and a block algebra is defined on each such clique has the amalgamation property. There is no need for this construction when $q = 3$.

Construction 4.2. For $q > 3$ a prime power, fix μ with $\mu(\boldsymbol{\alpha}) = q - 2$ and V as Definition 4.1. We transform an $A \in \boldsymbol{K}^q_\mu$ to an $A' \in \boldsymbol{K}'_\mu$. First, there is a canonical extension of A to $\check{A} \in \boldsymbol{K}^q_\mu$. Namely extend each clique of length at least 3 in A to have length q; but with no new intersections. Now expand $\check{A}$ to a τ'-structure by imposing on each line ℓ a copy of $F_2(V)$ with graph $H \restriction \ell$. Call this expansion A'.

[14]This image will be a substructure of a structure in $\boldsymbol{K}'_\mu$ but rarely itself a member of $\boldsymbol{K}'_\mu$.

We have in fact defined a finite family of possible expansions of A (depending on the interaction of H and R). The set of possible expansions of each $A \in \boldsymbol{K}_\mu$ as A varies through $\boldsymbol{K}^q_\mu$ is denoted $\boldsymbol{K}'_{\mu,V}$.

For any such A', let $\delta'(A') = \delta(A' \restriction \tau) = \delta(A)$. Note that each non-trivial line in A' has q elements. We denote by $\boldsymbol{\alpha}'$ the isomorphism type $(a_1, a_2, b_1 \ldots b_{k-2})$ of full line over two points; it is good with respect to $\boldsymbol{K}'_{\mu,V}$. (At various places in the definitions we must replace 'substructure' by 'substructure in $\boldsymbol{K}^q_\mu$.) This will not disturb the value of δ because δ of any line segment is 2.

Note that, except for $\boldsymbol{\alpha}'$, each good pair $\boldsymbol{\gamma} = A/B$ in $\boldsymbol{K}_\mu$ has generated a finite number of distinct good pairs in $\boldsymbol{K}'_\mu$. As, the various copies have the same reduct to τ but differ in their quasigroup structure. With this framework in hand we can complete the proof of Theorem 4.3. We show how to vary the proofs of the crucial results 5.11 and 5.15 from [BP20] for this result.

Theorem 4.3. *For each q and each $\mu \in \mathcal{U}$ each of the T_μ in Theorem 0.1 with line length $k = q = p^n$ (for some n) and certain varieties (block algebras) of quasigroups V, there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that defines a strongly minimal q-Steiner system.*

Proof. We can construct a generic, provided we prove amalgamation for $\boldsymbol{K}'_{\mu,V}$. We now show that the amalgamation for the τ-class, as in Lemma 5.11 and Lemma 5.15 of [BP20] yields an amalgamation for τ'. Consider a triple D', E', F' in $\boldsymbol{K}'_{\mu,V}$ as in Lemma 5.15 of the earlier paper. That is, $D' \subseteq F$ and E' is 0-primitive over D'. Since E' is primitive over D', although there may be a line contained in the disjoint amalgam G' with two points in each of D and $F - D$, each line that contains 2 points in $E - D$ can contain at most one from D. Thus, there is no issue with defining the relation H on the disjoint amalgamation. If μ' requires some identification for some (B', C'), just as in the original, it is because the (relational) τ'-structure $B'C'$ is $D'E'$ and (Note the 'further' in [BP20, Lemma 5.10].) there is a copy of E' over B' in F'. Now the strong minimality of the generic follows exactly as in Lemmas 5.21 and 5.23 of [BP20] and we have proved Theorem 4.3. ∎

Notation 4.4. *For each $\mu \in \mathcal{U}$ with $\mu(\boldsymbol{\alpha}) = q - 2$, and each $(2, q)$-variety V, we denote by $\mathcal{G}'_{\mu',V}$ the strongly minimal τ' generic structure constructed in Theorem 4.3. Its reduct to $\{H\}$ is a strongly minimal block algebra. The theory of that reduct is essentially $T_{\mu',V}$ since R is definable in that reduct by $R(u, v, w)$ if and only $\bigvee_{\sigma(u,v)} H(u, v, \sigma(u, v))$, where is $\sigma(u, v)$ is an existential formula formed by translating a term in $F_2(V)$ into a relational formula).*

QUESTION 4.5. Necessarily in the construction given, a τ good pair (B, C) in the reduct of a model of $T_{\mu',V}$ will have many (but finitely) more copies of C over B than $\mu(B, C)$. Thus, $T_{\mu',V} \restriction \tau$ is not T_μ.

Is it possible to characterize those μ such that T_μ can be interpreted in a quasigroup? We guaranteed that each 2-generated subalgebra is $F_2(V)$ and V is a Mikado variety (in particular, determined by 2-variable equations, each quasi-group Q_M is in V. This is enough to show the full structure is a quasigroup. But different varieties of quasigroups may have the same free algebra on two generators. Construction 4.2 depends on both the original $\boldsymbol{K}^q_\mu$ and $F_2(V)$. *How many varieties can arise from the same $F_2(V)$? There are two variants on this question. One is,* *'how many varieties of quasigroup can have the same free algebra on two generators?'. The second asks only of varieties $V(Q,H)$ that arise from a theory T_μ as in Construction 4.2.*

What varieties do the $\mathcal{G}_{\mu',V}$ generate? *Immediately from known results each such variety satisfies strong properties.*

Corollary 4.6. *For each $T_{\mu',V}$ with prime power line length, any $M \models T_\mu$, the reduct to $*$ is in a variety (that is congruence permutable, regular and uniform* [Qua76, Theorem 3.1] *or* [GW75, Corollary 2.4] *but not residually small* [BM88, Corollary 8].

QUESTION 4.7. *Every finite algebra in a $(2,q)$ has a finite decomposition into directly irreducible algebras (*[GW75, Corollary 2.4]*.* Are there any similar results for infinite strongly minimal block algebras?

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