Quaternion Matrices

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December 10, 2020

Outline

In this talk, I will introduce:

- the definition of quaternions
- some properties of matrices of quaternions
- algorithm of finding eigenvalues and eigenvectors of quaternion matrices

Quaternions

Definition

Denote the fields of the complex and real numbers by $\mathbb{C}$ and $\mathbb{R}$ respectively. Define

$$\mathbb{H} = \left\{ a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1; a_0, a_1, a_2, a_3 \in \mathbb{R} \right\},$$

which is a 4-dimensional associative algebra over $\mathbb{R}$ with an ordered basis, denoted by $\mathbf{1}, \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ and the multiplication table:

x	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\mathbf{1}$	1	i	j	k
$\boldsymbol{i}$	i	-1	k	$-j$
$\boldsymbol{j}$	j	$-k$	-1	i
$\boldsymbol{k}$	k	j	$-i$	-1

.

Quaternions

In other words, a real quaternion, simply called quaternion, is a vector

$$x = x_0\mathbf{e} + x_i\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$$

with coefficients $x_0, x_1, x_2, x_3 \in \mathbb{R}$. Besides the addition and the scalar multiplication of the vector space $\mathbb{H}$ over $\mathbb{R}$, the product of any two of the quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ can be computed by using the table, i.e.

$$\begin{gathered} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \end{gathered}$$

Quaternions

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$$\begin{gathered}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.\end{gathered}$$

For any nonzero quaternion $x = x_0 + x_i\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in$, we have

$$x^{-1} = \frac{x^*}{|x|^2} = \frac{1}{x_0^2 + x_1^2 + x_2^2 + x_3^2}(x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}).$$

Therefore, $\mathbb{H}$ is indeed a division ring.

Quaternions

Definition

Two quaternions x and y are said to be similar (or equivalent) if there exists a non-zero quaternion u such that $u^{-1}xu = y$, denoted by $x \sim y$.

Remark

It is obvious that $\sim$ is an equivalent relation and we will denote the equivalence class containing x by $[x]$.

Quaternions

Lemma (Fuzhen, 1997)

If $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, *then* x *and* $y = x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}\mathbf{i}$ *are similar, namely,* $x \in \left[x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}\mathbf{i}\right]$.

Proof.

If x is a complex number, then either $x = y$ or $x = \bar{y}$. Since we have $x\mathbf{j} = \mathbf{j}\bar{x}$, x and y are always similar. Otherwise, we can assume that $x_2^2 + x_3^2 \neq 0$ and set

$$u = \left(\sqrt{x_1^2 + x_2^2 + x_3^2} + x_1\right) - x_3\mathbf{j} + x_2\mathbf{k},$$

which is a non-zero number and thus invertible. Direct computation shows that$uy = xu$.

Therefore, we have $x = uyu^{-1}$, which shows that x is similar to y. ☐

The previous lemma allows us to transfer all the "weight" from the imaginary part onto the **i**-coordinate without changing the equivalent class, and therefore yields the following theorem and definition.

Theorem (Brenner, 1951; Au-Yeung, 1984)

. Let $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ *and* $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ *be quaternions. Then* x *and* y *are similar if and only if* $x_0 = y_0$ *and* $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$*, i.e.* $\operatorname{Re} x = \operatorname{Re} y$ *and* $|\operatorname{Im} x| = |\operatorname{Im} y|$*.*

Definition

For any given quaterninon $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$, the complex number $x = x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}\mathbf{i}$ is call the **principal number** of the class of $[x]$, denoted by $c(\lambda)$.

Quaternion Matrices

Because quaternions are non-commutative, sometimes it's helpful to think of them as ordered pairs of complex numbers. For any given quaternion

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$

where x_0, x_1, x_2 and x_3 are real numbers, we can rewrite the expression as

$$x = x_0 + x_1\mathbf{i} + (x_2 + x_3\mathbf{i})\,\mathbf{j},$$

and denote

$$c_1 = x_0 + x_1\mathbf{i} \;\text{ and } c_2 = x_2 + x_3\mathbf{i}.$$

It's easy to see that there exists a unique ordered pair $(c_1, c_2) \in \mathbb{C}^2$ such that $x = c_1 + c_2\mathbf{j}$.

Similarly, we see that any $A \in M_n(\mathbb{H})$ can be uniquely expressed in the form

$$A = A_1 + A_2\mathbf{j}$$

where $A_1, A_2 \in M_n(\mathbb{C})$. Therefore, we can define an injective map $\chi : M_n(\mathbb{H}) \longrightarrow M_{2n}(\mathbb{C})$ by

$$\chi(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}.$$

for all $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$, and call $\chi(A)$ the **complex adjoint matrix** of the quaternion matrix A, denoted by χ_A.

Quaternion Matrices

Remark

An alternative way to define quaternions is to consider the subset of the ring $M_2(\mathbb{C})$ of 2×2 matrices with complex number entries:

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ -\overline{a_2} & \overline{a_1} \end{bmatrix} : c_1, c_2 \in \mathbb{C} \right\}.$$

Thereofre, the **complex adjoint matrix** is a generalization of this definition.

For any quaternion matrices $A, B \in M_n(\mathbb{H})$, we have

$$\begin{aligned} AB &= (A_1 + A_2\mathbf{j})\,(B_1 + B_2\mathbf{j}) \\ &= \left(A_1B_1 - A_2\overline{B_2}\right) + \left(A_1B_2 + A_2\overline{B_1}\right)\mathbf{j} \end{aligned}$$

and

$$\begin{aligned} \chi(AB) &= \begin{bmatrix} A_1B_1 - A_2\overline{B_2} & A_1B_2 + A_2\overline{B_1} \\ -\overline{A_1B_2} - \overline{A_2}B_1 & \overline{A_1B_1} - \overline{A_2}B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \\ &= \chi(A)\chi(B), \end{aligned}$$

which means χ preserves matrix multiplication. Moreover, it is not difficult to observe that χ also preserves addition and (real) scalar multiplication. Therefore, χ is an injective $\mathbb{R}$-algebra homomorphism from $M_n(\mathbb{H})$ to $M_{2n}(\mathbb{C})$.

Quaternion Matrices

Theorem

Let $A, B \in M_n(\mathbb{H})$ *and* $a \in \mathbb{R}$. *Then*

(1) $\chi(I_n) = I_{2n}$;

(2) $\chi(aA) = a\chi(A)$;

(3) $\chi(A + B) = \chi(A) + \chi(B)$;

(4) $\chi(AB) = \chi(A)\chi(B)$;

(5) $\chi(A^{-1}) = \chi(A)^{-1}$ *if* A^{-1} *exists.*

With this tool, we can easily answer a common question that is often asked:

Question

Let $A, B \in M_n(\mathbb{H})$. If $AB = I$, the $n \times n$ identity matrix, is it true that $BA = I$?

Quaternion Matrices

Question

Let $A, B \in M_n(\mathbb{H})$. If $AB = I$, the $n \times n$ identity matrix, is it true that $BA = I$?

Proof.

Yes. If $AB = I_n$, then we can apply χ on both sides to have

$$\chi(A)\chi(B) = \chi(AB) = \chi(I_n) = I_{2n}.$$

Notice that both $\chi(A)$ and $\chi(B)$ are complex matrices, we have

$$\chi(BA) = \chi(B)\chi(A) = I_{2n} = \chi(I_n),$$

which implies $BA = I_n$ by the fact that χ is injective. ☐

Quaternion Matrices

Theorem (Fuzhen, 1997)

Let $A \in M_n(\mathbb{H})$*. Then the following are equivalent:*

(1) A *is invertible;*

(2) $Ax = 0$ *has a unique solution* 0*.*

(3) χ_A *is invertible.*

Proof.

((1) $\Longrightarrow$ (2)) Trivial.
((2) $\Longrightarrow$ (3)) Suppose $Ax = 0$ has a unique soluton 0. Denote

$$A = A_1 + A_2\mathbf{j} \text{ and } x = x_1 + x_2\mathbf{j}$$

where $A_1, A_2 \in M_n(\mathbb{C})$ and $x_1, x_2 \in \mathbb{C}^n$. Apply χ on both sides of $Ax = 0$, we have

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ -\overline{x_2} & \overline{x_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which deduces that the equation

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix} = 0$$

has a unique solution $x_1 = x_2 = 0$. Therefore, $\chi(A)$ is invertible.

Quaternion Matrices

((3) $\Longrightarrow$ (1)) Suppose $\chi(A)$ is invertible. Then there exists a complex matrix $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in M_{2n}(\mathbb{C})$ such that

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

We can then observe that this implies

$$\begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

Since χ is injective, we have that $B = B_1 + B_2\mathbf{j}$ is the inverse of A, i.e., A is invertible.

Eigenvalues and Eigenvectors

We now can turn our attention to the eigenvalues of quaternion matrices. Since left and right scalar multiplications are different, we need to treat $Ax = \lambda x$ and $Ax = x\lambda$ separately.

Definition

([Liping, 2001].) Given $A \in \mathbb{H}^{n\times n}, \lambda \in \mathbb{H}$ is called a left eigenvalue of A if $Av = \lambda v$ for some nonzero $v \in \mathbb{H}^n$. The set of distinct left eigenvalues is called the left spectrum of A, denoted $\sigma_l(A)$.

Definition

([Liping, 2001].) Given $A \in \mathbb{H}^{n\times n}, \lambda \in \mathbb{H}$ is called a right eigenvalue of A if $Av = v\lambda$ for some nonzero $v \in \mathbb{H}^n$. The set of distinct right eigenvalues is called the right spectrum of A, denoted $\sigma_r(A)$.

Eigenvalues and Eigenvectors

Example

Let $A = \begin{bmatrix} i+j & k \\ 0 & j \end{bmatrix}$ and $v = \begin{bmatrix} i \\ k \end{bmatrix}$. Then we have

$$Av = \begin{bmatrix} i+j & k \\ 0 & j \end{bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} = \mathbf{j} \begin{bmatrix} i \\ k \end{bmatrix} = \mathbf{j}v,$$

Therefore, $\mathbf{j}$ is a left eigenvalue of A. However, since this does not imply $Av = v\mathbf{j}$, $\mathbf{j}$ is not necessarily a right eigenvalue of A.

Right Eigenvalues

Lemma

If $\lambda \in \mathbb{H}$ is a right eigenvalue of $A \in M_n(\mathbb{H})$, then so is its principal number $c(\lambda)$.

Proof.

If λ is a right eigenvalue of $A \in M_n(\mathbb{H})$, then there exists $0 \neq p \in \mathbb{H}$ such that the principal of λ can be expressed as $x = p^{-1}\lambda p$. Suppose v is a eigenvector of A associated with the right eigenvalue λ, then we can see that

$$Av = v\lambda$$

is equivalent to

$$A(vp) = (vp)(p^{-1}\lambda p),$$

which means the principal number $c(\lambda) = p^{-1}\lambda p$ is also a right eigenvalue of A. ☐

Therefore, finding all right eigenvalues of a matrix reduces to finding all of its complex right eigenvalues. This motivates us to investigate the complex adjoint matrix $\chi(A)$ again:

Consider the function $\rho : \mathbb{H}^n \to \mathbb{C}^{2n}$ by

$$\rho\left(\begin{bmatrix} c_1^{(1)} + c_2^{(1)}j \\ c_1^{(2)} + c_2^{(2)}j \\ \vdots \\ c_1^{(n)} + c_2^{(n)}j \end{bmatrix}\right) = \begin{bmatrix} c_1^{(1)} \\ \vdots \\ c_1^{(n)} \\ -\overline{c_2}^{(1)} \\ \vdots \\ -\overline{c_2}^{(n)} \end{bmatrix},$$

i.e.,

$$\rho(v_1 + v_2\mathbf{j}) = \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} = \chi(v)\mathbf{e_1}.$$

This idea helps us to spread a quaternion vector open and turn it into a complex vector.

Definition

For any vector $v \in \mathbb{H}^n$, the vector $\rho(v) := \chi(v)\mathbf{e_1}$ is called the **unfolding** of v.

Definition

For any nonzero vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ where $w_1, w_2 \in \mathbb{C}^n$, the **companion vector** of w is defined by $w^c := \begin{bmatrix} -\overline{w_2} \\ \overline{w_1} \end{bmatrix}$.

Right Eigenvalues

Remark

Let $v \in \mathbb{H}^n$. Then the vectors $\chi(v)\mathbf{e_1}$ is the companion vector of $\chi(v)\mathbf{e_2}$, and vice versa. Moreover, there are similar notions about the companion vector in [Douglas, 2003]. and [Terry, 2012]. Please note that in [Tongsong, 2004], the companion vector is defined in a different way where the entries are permutated.

Lemma

If v and v^c be a pair of companion vectors, then they are linear independent.

Proof.

Suppose nonzero vectors $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $w^c := \begin{bmatrix} -\overline{w_2} \\ \overline{w_1} \end{bmatrix}$,where $w_1, w_2 \in \mathbb{C}^n$, are linearly dependent. Then we can assume without loss of generality that $w = kw^c$ for some $0 \neq k \in \mathbb{C}$, i.e.,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = k \begin{bmatrix} -\overline{w_2} \\ \overline{w_1} \end{bmatrix},$$

which implies

$$w_1 = -k\overline{w_2} \tag{1}$$

and

$$w_2 = k\overline{w_1}. \tag{2}$$

Substitute (2) into (1), we have $w_1 = -|k|^2 w_1$, which deduces

$$(1 + |k|^2)w_1 = 0.$$

Since $1 + |k|^2 > 0$, it forces $w_1 = 0$. We have a contradiction. □

Right Eigenvalues

Theorem

Let $A \in M_n(\mathbb{H}), w \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$. Then the following are equivalent:

(1) *w is a eigenvector of $\chi(A)$ associated with the eigenvalue λ, i.e.,*

$$\chi(A)w = w\lambda,$$

(2) *w^c is eigenvector of $\chi(A)$ associated with the eigenvalue $\overline{\lambda}$, i.e.,*

$$\chi(A)w^c = w^c\overline{\lambda}.$$

(3) *There exists a vector $v \in \mathbb{H}^n$ such that $w = \rho(v)$ and v is eigenvector of A associated with the right eigenvalue λ, i.e.,*

$$Av = v\lambda.$$

Proof.

Denote $A = A_1 + A_2\mathbf{j}$ and $v = v_1 + v_2\mathbf{j}$ where $A_1, A_2 \in M_n(\mathbb{C})$ and $v_1, v_2 \in \mathbb{C}^n$. If v is an eigenvector of A associated with the right eigenvalue $\lambda \in \mathbb{C}$, i.e.,

$$Av = v\lambda. \tag{3}$$

Apply the χ on both sides, we have

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ -\overline{v_2} & \overline{v_1} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ -\overline{v_2} & \overline{v_1} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}, \tag{4}$$

which deduces

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} = \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} \lambda \tag{5}$$

and

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_2 \\ \overline{v_1} \end{bmatrix} = \begin{bmatrix} v_2 \\ \overline{v_1} \end{bmatrix} \overline{\lambda}. \tag{6}$$

We can observe that equation (5) and (6) are equivalent, so they are both equivalent to (4). Besides, since χ is injective, (4) is equivalent to (3). Moreover, in (5) the complex vector $w = \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix}$, whose companion vector is the vector $w^c = \begin{bmatrix} v_2 \\ \overline{v_1} \end{bmatrix}$ in (6), is exactly the unfolding of the quaternion vector v. ■

Right Eigenvalues

Therefore, we can see that each right complex eigenvalue of A is one-to-one corresponding to a pair of eigenvalues of $\chi(A)$.

Moreover, since each pair of the companion vectors w and w_c are always linearly independent, we can see that all real eigenvalues of $\chi(A)$ have algebraic multiplicity even number times and all non-real eigenvalues of $\chi(A)$ appear in conjugate pairs.

Based on the discussion, we immediately have the following results:

Right Eigenvalues

Theorem

([Lee, 1949].). Let A and B be n × n complex matrices. Then every real eigenvalue (if any) of the matrix

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

appears an even number of times, and the complex eigenvalues of that matrix appear in conjugate pairs.

Corollary

([Brenner, 1951], [Lee, 1949]). Any n × n quaternion matrix A has exactly n right eigenvalues which are complex numbers with non-negative imaginary parts.

Right Eigenvalues

Now, we have enough foundation to show the this important result.

Theorem

Let $A \in M_n(\mathbb{H})$. Then A is diagonalizable over $\mathbb{H}$ if and only if $\chi(A)$ is diagonalizable over $\mathbb{C}$.

Proof.

($\Longrightarrow$) Suppose A is diagonalizable over $\mathbb{H}$. Then there exists an invertible matrix $Q \in M_n(\mathbb{H})$ such that

$$Q^{-1}AQ = diag(\lambda_1, ..., \lambda_n).$$

Denote $Q = [v_1, v_2, .., v_n]$ and the principal numbers $c(\lambda_i) = p_i^{-1}\lambda p_i$ for some nonzero $p \in \mathbb{H}$, $i = 1, 2, ..., n$. Define $P = diag(p_1, p_2, ..., p_n)$ and $S = QP$, then S is invertible, since both P and Q are invertible.

Then we have

$$\begin{aligned} S^{-1}AS &= P^{-1}(Q^{-1}AQ)P \\ &= P^{-1}diag\left(\lambda_1, ..., \lambda_n\right)P \\ &= diag\left(p_1^{-1}\lambda_1 p_1, ..., p_n^{-1}\lambda_n p_n\right) \\ &= diag\left(c(\lambda_1), ..., c(\lambda_n)\right). \end{aligned}$$

Apply χ on both side, we have

$$\chi(S)^{-1}\chi(A)\chi(S) = diag\left(c(\lambda_1), ..., c(\lambda_n), \overline{c(\lambda_1)}, ..., \overline{c(\lambda_n)}\right),$$

which shows that $\chi(A)$ is diagonalizable over $\mathbb{C}$.

Right Eigenvalues

$(\Longleftarrow)$ Suppose $\chi(A)$ is diagonalizable over $\mathbb{C}$. By the previous theorem , the real eigenvalues of $\chi(A)$ have multiplicity even number times and the non-real eigenvalues of $\chi(A)$ appear in conjuate pairs. Therefore, we can denote

$$\sigma\left(\chi(A)\right) = \left\{\lambda_1, \overline{\lambda_1}, ..., \lambda_n, \overline{\lambda_n}\right\}.$$

Then by the previous theorem, we can always select a pair of companion vectors (v_i, v_i^c) for each pair of the eigenvalues $(\lambda_i, \overline{\lambda_i})$, $i = 1, 2, ..., n$, in the spectrum of $\chi(A)$. Now, we can construct an invertible complex matrix by letting

$$P = [v_1, v_2, ..., v_n, v_1^c, v_2^c, ..., v_n^c],$$

Right Eigenvalues

and we have

$$P^{-1}\chi(A)P = diag\left(\lambda_1, \lambda_2, ..., \lambda_n, \overline{\lambda_1}, \overline{\lambda_2}, ..., \overline{\lambda_n}\right). \tag{7}$$

Obverse that $P \in Im(\chi)$, thus $\chi^{-1}(P)$ is defined. In fact, we have

$$\chi^{-1}(P) = \left[\rho^{-1}(v_1), \rho^{-1}(v_2), ..., \rho^{-1}(v_n)\right].$$

Now, we can apply $\chi{-}1$ on both sides of equation (7) to have

$$\chi^{-1}(P)^{-1}A\chi^{-1}(P) = diag\,(\lambda_1, \lambda_2, ..., \lambda_n),$$

which shows that A is diagonalizable over $\mathbb{H}$.

Corollary

Let $A \in M_n(\mathbb{C})$. Then A is diagonalizable over $\mathbb{H}$ if and only if A is diagonalizable over $\mathbb{C}$.

Right Eigenvalues

Example

Let $A = \begin{bmatrix} 1-j+k & j+k \\ 1+i-j-k & i-j+k \end{bmatrix}$. Then we have

$$\chi(A) = \begin{bmatrix} 1 & 0 & -1+i & 1+i \\ 1+i & i & -1-i & -1+i \\ 1+i & -1+i & 1 & 0 \\ 1-i & 1+i & 1-i & -i \end{bmatrix},$$

whose characteristic polynomial is

$$f(\lambda) = \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1.$$

Therefore, we have $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \mathbf{i}$ and $\lambda_4 = -\mathbf{i}$.

For $\lambda_1 = \lambda_2 = 1$, we have 2 linearly independent eigenvectors (a companion pair)

$$w_1 = \begin{bmatrix} -\mathbf{i} \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad w_1^c = \begin{bmatrix} 0 \\ 0 \\ \mathbf{i} \\ 1 \end{bmatrix}.$$

For $\lambda_3 = \mathbf{i}$ and $\lambda_4 = -\mathbf{i}$, we have 2 linearly independent eigenvectors (a companion pair)

$$w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -\mathbf{i} \end{bmatrix} \quad \text{and} \quad w_2^c = \begin{bmatrix} -1 \\ -\mathbf{i} \\ 0 \\ 1 \end{bmatrix}.$$

Let $P = [w_1, w_2, w_1^c, w_2^c,]$, we have

$$P^{-1}\chi(A)P = diag(1, 1, i, -i).$$

Recover the quaternion vectors by

$$v_1 = \chi^{-1}([w_1, w_1^c]) = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

and

$$v_2 = \chi^{-1}([w_2, w_2^c]) = \begin{bmatrix} -j \\ 1-k \end{bmatrix},$$

and let $Q = [v_1, v_2]$, we can check

$$\begin{bmatrix} 1-j+k & j+k \\ 1+i-j-k & i-j+k \end{bmatrix} \begin{bmatrix} -i & -j \\ 1 & 1-k \end{bmatrix} = \begin{bmatrix} -i & -j \\ 1 & 1-k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},$$

which deduces

$$Q^{-1}AQ = diag(1, i).$$

Therefore, the right spectrum of A is

$$\sigma_r(A) = \left\{ w^{-1}\mathbf{i}w \mid w \in \mathbb{H}, w \neq 0 \right\} \cup \{1\}.$$

Right Eigenvalues

Example [Tongsong, 2004].

Let $A = \begin{bmatrix} i & 1+j \\ -1+j & -k \end{bmatrix}$. Then we have

$$\chi(A) = \begin{bmatrix} i & 1 & 0 & 1 \\ -1 & 0 & 1 & -i \\ 0 & -1 & -i & 1 \\ -1 & -i & -1 & 0 \end{bmatrix},$$

whose characteristic polynomial is

$$f(\lambda) = (\lambda + 3)^2 = (\lambda + \sqrt{3}\mathbf{i})^2(\lambda - \sqrt{3}\mathbf{i})^2.$$

Therefore, we have $\lambda_1 = \sqrt{3}\mathbf{i}$ and $\lambda_2 = -\sqrt{3}\mathbf{i}$.

For $\lambda_1 = \sqrt{3}$, we have 2 linearly independent eigenvectors

$$w_1 = \begin{bmatrix} 1+\sqrt{3} \\ 2\mathbf{i} \\ 1-\sqrt{3} \\ 0 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 2\mathbf{i} \\ 1-\sqrt{3} \\ 0 \\ 1-\sqrt{3} \end{bmatrix}.$$

Recover the quaternion vectors by

$$v_1 = \chi^{-1}([w_1, w_1^c]) = \begin{bmatrix} (1+\sqrt{3}) - (1-\sqrt{3})\mathbf{j} \\ 2\mathbf{i} \end{bmatrix}$$

and

$$v_2 = \chi^{-1}([w_2, w_2^c]) = \begin{bmatrix} 2\mathbf{i} \\ (1-\sqrt{3}) - (1-\sqrt{3})\mathbf{j} \end{bmatrix}.$$

Right Eigenvalues

Let $Q = [v_1, v_2]$, we have

$$Q^{-1}AQ = diag(\sqrt{3}i, \sqrt{3}i).$$

Therefore, the right spectrum of A is

$$\sigma_r(A) = \left\{ w^{-1}\sqrt{3}\mathbf{i}w \mid w \in \mathbb{H}, w \neq 0 \right\}.$$

Left Eigenvalues

Now, let us try to investigate the left eigenvalues of the quaternion matrices.

Denote $A = A_1 + A_2\mathbf{j}$ and $v = v_1 + v_2\mathbf{j}$ where $A_1, A_2 \in M_n(\mathbb{C})$ and $v_1, v_2 \in \mathbb{C}^n$. If v is an eigenvector of A associated with the left eigenvalue $\lambda \in \mathbb{C}$, i.e.,

$$Av = \lambda v, \tag{8}$$

then we can apply the χ on both sides to have

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ -\overline{v_2} & \overline{v_1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -\overline{\lambda}_2 & \overline{\lambda}_1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ -\overline{v_2} & \overline{v_1} \end{bmatrix}, \tag{9}$$

which is equivalent to

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -\overline{\lambda}_2 & \overline{\lambda}_1 \end{bmatrix} \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} \tag{10}$$

and

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} v_2 \\ \overline{v_1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -\overline{\lambda}_2 & \overline{\lambda}_1 \end{bmatrix} \begin{bmatrix} v_2 \\ \overline{v_1} \end{bmatrix}. \tag{11}$$

We can observe that equation (10) and (11) are equivalent, so they are both equivalent to (9). Besides, since χ is injective, (9) is equivalent to (8). Therefore, solving (8) is equivalent to solving (10). Because (10) is also equivalent to

$$\begin{bmatrix} (A_1 - \lambda_1 I_n) & (A_2 - \lambda_2 I_n) \\ -\left(\overline{A_2} - \overline{\lambda}_2 I_n\right) & \left(\overline{A_1} - \overline{\lambda}_1 I_n\right) \end{bmatrix} \begin{bmatrix} v_1 \\ -\overline{v_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

Left Eigenvalues

Therefore, A has a left eigenvalue if and only if there exist complex numbers λ_1 and λ_2 such that

$$det\left(\begin{bmatrix} (A_1 - \lambda_1 I_n) & (A_2 - \lambda_2 I_n) \\ -\left(\overline{A_2} - \overline{\lambda}_2 I_n\right) & \left(\overline{A_1} - \overline{\lambda}_1 I_n\right) \end{bmatrix}\right) = 0. \tag{13}$$

For any given A_1 and A_2, how can we find λ_1 and λ_2? Unfortunately, this is still unknown. In general, the best we have now is the fact that such λ_1 and λ_2 always exist.

Left Eigenvalues

Theorem (Wood, 1985)

Every $n \times n$ quaternion matrix has at least one left eigenvalue in $\mathbb{H}$.

Since the proof the the theorem basically uses algebraic topology (see [Fuzhen, 1997]), we do not include it here.

Left Eigenvalues

However, algorithms for left eigenvalues of 2 × 2 quaternion matrices has been established.

We will include the following theorem without proof.

Theorem (Liping, 2001)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternion matrix.*

(1) *If* $bc = 0$, *then* $\sigma_l(A) = \{a, d\}$.

(2) *If* $bc \neq 0$, *then* $\sigma_l(A) = \big\{a + b\lambda : \lambda^2 + b^{-1}(a-d)\lambda - b^{-1}c = 0\big\}$.

Therefore, solving the left spectrum of a quaternion matrix reduces to solving a quadratic equation over quaternions.

Left Eigenvalues

Theorem (Liping, 2000)

The solutions of the quadratic equation $x^2 + bx + c = 0$ *can be obtained by formulas according to the following cases:*

1. If $b, c, \in \mathbb{R}$ *and* $b^2 < 4c$, *then*

$$x = \frac{1}{2}(-b + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k})$$

for all $\beta, \gamma, \delta \in \mathbb{R}$ *with* $\beta^2 + \gamma^2 + \delta^2 = 4c - b^2$.

2. If $b, c, \in \mathbb{R}$ *and* $b^2 \geqslant 4c$, *then*

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Left Eigenvalues

3. If $b \in \mathbb{R}$ and $c \notin \mathbb{R}$, then

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k}$$

where $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $c_i \in \mathbb{R}$, and

$$\rho = \sqrt{\frac{b^2 - 4c_0 + \sqrt{\left(b^2 - 4c_0\right)^2 + 16\left(c_1^2 + c_2^2 + c_3^2\right)}}{2}}.$$

4. If $b \notin \mathbb{R}$, then

$$x = \frac{-\operatorname{Re} b}{2} - \left(b' + T\right)^{-1}\left(c' - N\right)$$

where

$$b' = b - \operatorname{Re} b = \operatorname{Im} b, \quad c' = c - \frac{\operatorname{Re} b}{2}\left(b - \frac{\operatorname{Re} b}{2}\right)$$

and (T, N) is chosen as
(a) $T = 0, N = \left(B \pm \sqrt{B^2 - 4E}\right)/2$ provided that $D = 0, B^2 \geqslant 4E$.
(b) $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E$.
(c) $T = \pm\sqrt{z}, N = \left(T^3 + BT + D\right)/2T$ provided that $D \neq 0$ where z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + \left(B^2 - 4E\right)z - D^2$, where $B = |b'|^2 + 2\operatorname{Re} c', E = |c'|^2$, and $D = 2\operatorname{Re} \overline{b'}c'$.

Left Eigenvalues

Example

Let

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{j} & 1 \end{bmatrix}$$

Then

$$\begin{aligned} \sigma_l(A) &= \left\{ \mathbf{i}\lambda : \lambda^2 + \mathbf{i}\lambda + \mathbf{k} = 0 \right\} \\ &= \left\{ \mathbf{i}\lambda : \lambda = \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}), \frac{1}{2}(-1 - \mathbf{i} - \mathbf{j} + \mathbf{k}) \right\} \\ &= \left\{ \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} - \mathbf{k}), \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}) \right\}. \end{aligned}$$

Left Eigenvalues

Example

Let

$$A = \begin{bmatrix} 2 & \mathbf{i} \\ -\mathbf{i} & 2 \end{bmatrix}$$

Then

$$\begin{aligned} \sigma_l(A) &= \left\{ 2 + \mathbf{i}\lambda : \lambda^2 + 1 = 0 \right\} \\ &= \left\{ 2 - \mathbf{i}\lambda : \lambda = \frac{1}{2}(\beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}), \beta^2 + \gamma^2 + \delta^2 = 4 \right\} \\ &= \left\{ 2 - \beta\mathbf{i} - \delta\mathbf{j} + \gamma\mathbf{k} : \beta^2 + \gamma^2 + \delta^2 = 1 \right\}. \end{aligned}$$

That's all. Thank you!