Locally projective modules, Zimmermann-Huisgen's 'Property (A)', and the elementary dual

T. G. Kucera[1] Ph. Rothmaler[2]

[1]Department of Mathematics, University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

[2]Bronx Community College, City University of New York
New York, NY USA

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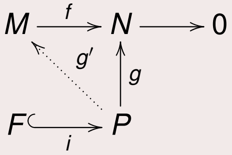
Preliminaries

Local projectivity

Definition

A module ${}_R P$ is *locally projective* iff all diagrams



with the top row exact and F finitely generated, can be completed as shown, so that $fg'i = fg$.

A characterization of projectivity

The following are equivalent:

1. ${}_RP$ is projective;
2. ${}_RP$ is a direct summand of a free left R-module;
3. **Dual Basis Theorem:**
   There are elements $\langle p_i \rangle_{i \in I}$ and homomorphisms $\langle f_i \rangle_{i \in I}$, $f_i : P \to R$, such that for every $p \in P$,
   1. $f_i(p) = 0$ for almost all $i \in I$;
   2. $p = \sum_{i \in I} f_i(p) p_i$.

Flat modules

Characterizations of flatness

The following are equivalent:

1. ${}_RF$ is flat;
2. The functor $\_\_\_ \otimes F$: Mod-$R \to$ **Ab** is exact;
3. For every finite $\mathbf{r} \in R$ and corresponding $\mathbf{x} \in F$ such that $\mathbf{r} \cdot \mathbf{x} = 0$, there is a (finite) matrix A over R and tuple $\mathbf{y} \in F$ such that $\mathbf{r}A = 0$ and $A\mathbf{y} = \mathbf{x}$.
4. (Rothmaler) For every left pp formula $\varphi(\mathbf{x})$, $\varphi[F] = \varphi[{}_RR]F$.

Free implies Projective implies Flat

Infinite matrices

Let I and J be non-empty sets; generally treated as index sets with no other implied structure.

An $I \times J$ matrix over R is a function $A : I \times J \to R$.

For each $i \in I$, the restriction ${}_iA$ of A to $\{ i \} \times J$ is called a *row* of A. Similarily we define the *columns* of A.

A is *row finite* if every row of A is 0 almost everywhere.
Similarly, *column finite*.

Similarly, row or column *vectors* of *elements* of R, of *variables*, of some R-module M.

Algebra of matrices

${}_{I}A^{J}$ is an $I \times J$ matrix.

Two matrices of the same shape may be multiplied by a scalar (element of R) or added, to yield a matrix of the same shape.

An infinite sum almost all of whose terms are 0 is treated as well-defined.

Two matrices ${}_{I}A^{J}$ and ${}_{J}B^{K}$ are *compatible for multiplication* if A is row finite or if B is column finite, or if more generally, for each $i \in I$ and each $k \in K$, $a_{ij}b_{jk} = 0$ for almost all $j \in J$; in which case we use the usual definition of matrix multiplication.

Systems of linear equations

Let A be a row-finite $I \times J$ matrix over R, $\mathbf{x}$ a $J \times 1$ matrix (column vector) of *variables*, and $\mathbf{b}$ an $I \times 1$ column vector of elements from an R-module ${}_R M$.

$$A\mathbf{x} = \mathbf{b}$$

is a *system of linear equations* in $\mathbf{x}$ over M.

A *solution* $A\mathbf{x} = \mathbf{b}$ is some $\mathbf{n}$, a $J \times 1$ column vector in some module ${}_R N \geq {}_R M$, such that in N, $A\mathbf{n} = \mathbf{b}$.

Solvability and Consistency

A system of linear equations over M is *solvable* ("semantically consistent") if it has a solution in some extension of M,.

[First year linear algebra]

$A\mathbf{x} = \mathbf{b}$ is *formally* (*syntactically*) *consistent* if for all ${}_1\mathbf{r}^I$ over R which are 0 almost everywhere, $\mathbf{r}A = \mathbf{0}$ implies $\mathbf{r}\mathbf{b} = 0$.

Theorem

$A\mathbf{x} = \mathbf{b}$ is solvable
iff
$A\mathbf{x} = \mathbf{b}$ is formally consistent.

Amongst the many characterizations of injectivity:

Theorem

A module ${}_R E$ is injective
iff
every formally consistent system of equations over E
has a solution in E .

Key point in the proof: using a system of equations to construct a homomorphism.

Lecture 2

…I don't know my left hand from my right hand

Property (A)

Reference

Birge Zimmermann-Huisgen, *Pure Submodules of direct products of free modules*, **Math. Ann. 224**, 233–245 (1976).

Property (A)

Let P_R be a right R-module.

(A) For all column-finite matrices ${}_I A^J$ over R and all I-rows $\mathbf{m} \in P$ such that $\mathbf{m}A = 0$, and all finite $I' \subset I$,

there is a finite k-row $\mathbf{x} \in P$ and a matrix ${}_k B^I$ over R such that

$$BA = 0 \text{ and } \mathbf{m}^{I'} = \mathbf{x}B^{I'} .$$

Recall: Flat modules

Characterizations of flatness

The following are equivalent:

1. F_R is flat;
2. For every finite $\mathbf{r} \in R$ and corresponding $\mathbf{x} \in F$ such that $\mathbf{xr} = 0$, there is a (finite) matrix A over R and tuple $\mathbf{y} \in F$ such that $A\mathbf{r} = 0$ and $\mathbf{yA} = \mathbf{x}$.

Recall: Local projectivity

Definition

A module ${}_RP$ is *locally projective* iff all diagrams

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \longrightarrow & 0 \\ & \nwarrow^{g'} & \uparrow{\scriptstyle g} & & \\ F & \xhookrightarrow[i]{} & P & & \end{array}$$

with the top row exact and F finitely generated,
can be completed as shown, so that $fg'i = gi$.

Characterizations of Locally Projective Modules

Theorem (B. Zimmermann-Huisgen)

The following are equivalent for a module P_R:

1. P is locally projective;
2. P has Property (A);
3. For each element $m \in P$, there are $x_1, \dots x_n \in P$ and homomorphisms $f_1, \dots, f_n : P \to R_R$ such that $m = \sum_j x_j f_j(m)$;
4. For each finite number of elements $m_1, \dots, m_k \in P$, there are $x_1, \dots x_n \in P$ and homomorphisms $f_1, \dots, f_n : P \to R_R$ such that for each i, $1 \leq i \leq k$,

$$m_i = \sum_j x_j f_j(m_i) .$$

Tidying up 'Property (A)': quantifiers

As stated, Property (A) has quantifiers scattered in leading and trailing positions. . . .

Property (A), restated

P_R has property (A) iff
  for all index sets I and J,
  for all ${}_{(I)}A^J$ over R,
  for all $\mathbf{m}^I \in P$,
  for all finite $I' \ll I$,

there are finite K, $\mathbf{x}^K \in P$, ${}_K B^I$ over R,

  such that $BA = \mathcal{O}$ and $\mathbf{m}^{I'} = \mathbf{x}B^{I'}$.

Tidying up 'Property (A)': families of properties

Property (A), restated one more time

P_R has property (A) iff
For all index sets I and J and all matrices ${}_{(I)}A^J$ over R,
P_R satisfies the following property $\left({}_{(I)}A^J\right)$:

$\left({}_{(I)}A^J\right)$ $\forall \mathbf{m}^I \in P$, for all $I' \ll I$,
there are finite K, $\mathbf{x}^K \in P$, ${}_K B^I$ over R with $BA = \mathcal{O}$
such that $\mathbf{m}^{I'} = \mathbf{x} B^{I'}$.

Tidying up ‘Property (A)’: $\left({}_{(I)}A^J\right)$

Expressing $\left({}_{(I)}A^J\right)$ as an infinitary implication

1. $\left({}_{(I)}A^J\right)$ is a property $Q(\overline{v})$ with variables $\overline{v}$ indexed by I of I-tuples $\mathbf{m} \in P$;
2. “for all $I' \ll I$” is a conjunction of properties indexed by the finite subsets of I,
3. the existential quantifiers can be written as disjunctions over certain index sets;
4. in particular, the rows of B all must be elements of the *left annihilator* $\mathcal{B}$ of A:

$$\mathcal{B} = \left\{ \mathbf{b}^{\mathbf{l}} : \mathbf{b}A = \mathcal{O} \right\}$$

Tidying up ‘Property (A)’: $\left({}_{(I)}A^J\right)$ (continued)

$$\left({}_{(I)}A^J\right) \; : \qquad \forall \overline{v}^I \left[\overline{v}A = \mathcal{O} \longrightarrow \bigwedge_{I' \ll I} \bigvee_{k \in \omega} \bigvee_{\overline{x}^k \in P} \bigvee_{{}_kB^I \in \mathcal{B}} (\overline{v}^{I'} = \overline{x}B^{I'}) \right]$$

The order of the disjunctions doesn’t matter, so “$\bigvee_{\overline{x}^k \in P}$” can return to an existential quantifier, and the other pair of disjunctions indexes a family closed upwards under sums, so we get:

$$\left({}_{(I)}A^J\right) \; : \qquad \forall \overline{v}^I \left[\overline{v}A = \mathcal{O} \longrightarrow \bigwedge_{I' \ll I} \sum_{{}_kB^I \in \mathcal{B}} \exists \overline{x}^k (\overline{v}^{I'} = \overline{x}B^{I'}) \right]$$

This is the universal closure of an implication between two generalized infinitary positive primitive formulas in the sense of my current work with Rothmaler; and falls into the form of a sentence which has an elementary dual.

a bunch of general stuff about elementary duality. . . .
October 20, 2020

Recall Elementary Duality (very informal summary)

- Elementary duality is a lattice anti-isomorphism between the lattice of pp formulas in the language of left R-modules and the lattice of pp-formulas in the language of right R-modules (up to logical equivalence).
- Elementary duality provides a categorical equivalence at the level of Shelah's "Imaginary Universe" between the category of left R-modules and the category of right R-modules.
- As a consequence of the pp-elimination of quantifiers for modules, a natural way of axiomatizing theories of modules (complete or otherwise) is by families of pp-implications: the universal closures of formulas of the form $\varphi \to \psi$, where φ and ψ are pp formulas.
- The elementary dual theory is axiomatized by the implications $\mathrm{D}\psi \to \mathrm{D}\varphi$.

Infinitary pp formulas

Fix an index set I for the (free) variables of infinitary pp formulas. In this context it is intended that I be an infinite set.

- There are two ways of expanding a finitary pp-formula $\varphi(\overline{x})$ to an I-formula:
  1. $\varphi^{\mathrm{E}} = \varphi(\overline{x}) \wedge \bigwedge_{v \notin \overline{x}} (v = v)$
  2. $\varphi^{\Omega} = \varphi(\overline{x}) \wedge \bigwedge_{v \notin \overline{x}} (v = 0)$
- Finitary pp formulas are closed under conjunction and sum in a natural way; infinitary analogues are only closed under conjunction "naturally", so we introduce a new infinitary operator Σ with semantics $M \models \Sigma_{j \in J} \varphi_j[\overline{a}]$ (where the φ_j are infinitary pp formulas) iff $\overline{a} \in \Sigma_{j \in J} \varphi_j[M]$, the sum in the sense of sum of subgroups.

Dualizable I-properties

Generalizations to the context of I-properties of results of Prest, Rothmaler and Ziegler.

Theorem (Kucera-Rothmaler)

1. A property P of I-tuples in a module M is dualizable iff it can be defined by an infinitary pp-formula of the form $\bigwedge_J \sum_{K_j} \varphi_{jk}^{\mathrm{E}}$ where each φ_{jk} is an ordinary finitary pp formula in some variables indexed from I.
2. A property P of I-tuples in a module M which are zero almost everywhere is dualizable iff it can be defined by an infinitary pp-formula of the form $\bigwedge_J \sum_{K_j} \varphi_{jk}^{\Omega}$ where each φ_{jk} is an ordinary finitary pp formula in some variables indexed from I.

Generalizations to the context of I-properties of results of Prest, Rothmaler and Ziegler. Formulas as on the previous slide.

Theorem (Kucera-Rothmaler)

Let $\mathcal{F}$ be the set of all choice functions on $\{ K_j : j \in J \}$.

1. $\mathsf{D}(\bigwedge_J \sum_{K_j} \varphi_{jk}^{\mathsf{E}}) = \sum_J \bigwedge_{K_j} (\mathsf{D}\varphi_{jk})^{\Omega} = \bigwedge_{f \in \mathcal{F}} \sum_{j \in J} (\mathsf{D}\varphi_{j,f(j)})^{\Omega}$.
2. $\mathsf{D}(\bigwedge_J \sum_{K_j} \varphi_{jk}^{\Omega}) = \sum_J \bigwedge_{K_j} (\mathsf{D}\varphi_{jk})^{\mathsf{E}} = \bigwedge_{f \in \mathcal{F}} \sum_{j \in J} (\mathsf{D}\varphi_{j,f(j)})^{\mathsf{E}}$

Reminder: Axiomatiation of Property (A)

P_R has property (A) iff
for all index sets I and J and all matrices ${}_{(I)}A^J$ over R,

$$\left({}_{(I)}A^J\right) \ : \qquad \forall \overline{v}^I \left[\overline{v}A = \mathcal{O} \longrightarrow \bigwedge_{I' \ll I} \sum_{{}_k B^I \in \mathcal{B}} \exists \overline{x}^k \left(\overline{v}^{I'} = \overline{x}B^{I'}\right) \right]$$

Dual of $\overline{v}A = \mathcal{O}$

- $\overline{v}A = \mathcal{O}$ is equivalent to $\bigwedge_{j \in J} \overline{v}A^j = 0$.

  Since A is column finite, $\overline{v}A^j$ is just an ordinary finite linear combination, which makes no assertion about the value of any v_i, i not in the support of $A^j = I_j \ll I$.
- $\overline{v}A^j = 0$ is thus essentially $((\overline{v}^{I_j})({}_{I_j}A^j) = 0)^{\mathrm{E}}$.
- The dual of this formula is $[\exists w({}_{I_j}\overline{v} = {}_{I_j}A^j w)]^{\Omega}$.
- The dual of $\overline{v}A = \mathcal{O}$ is

$$\sum_{j \in J} \exists w(\overline{v} = A^j w)$$

- The solution set in a left module N (that is, as a subset of $N^{(I)}$) is $AN^{(J)}$.

Dual of $\bigwedge_{l' \ll l} \sum_{{}_k B^l \in \mathcal{B}} \exists \overline{x}^k (\overline{v}^{l'} = \overline{x} B^{l'})$

- $\exists \overline{x}^k (\overline{v}^{l'} = \overline{x} B^{l'})$ is understood as $[\exists \overline{x}^k (\overline{v}^{l'} = \overline{x} B^{l'})]^{\mathsf{E}}$.
- This has dual $[{}_k B^{l'}{}_{l'} \overline{v} = 0]^{\Omega}$

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- The dual of the whole formula is then $\sum_{l' \ll l} \bigwedge_{{}_k B^l \in \mathcal{B}} [{}_k B^{l'}{}_{l'} \overline{v} = 0]^{\Omega}$

Dual of $\bigwedge_{l'\ll l}\sum_{_kB^l\in\mathcal{B}}\exists\overline{x}^k(\overline{v}^{l'}=\overline{x}B^{l'})$

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- The dual of the whole formula is then $\sum_{l'\ll l}\bigwedge_{_kB^l\in\mathcal{B}}[_kB^{l'}{}_{l'}\overline{v}=0]^{\Omega}$
- The conjunction can be evaluated "one row at a time" so is equivalent to $\bigwedge_{\left\{{}_1\overline{b}^l\,:\,\overline{b}A=0\right\}}(\overline{b}^{l'}{}_{l'}\overline{v}=0)^{\Omega}$

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- The conjunction can be evaluated "one row at a time" so is equivalent to $\bigwedge_{\left\{ {}_1 \overline{b}^l \,:\, \overline{b} A = 0 \right\}} (\overline{b}^{l'}{}_{l'} \overline{v} = 0)^{\Omega}$
- Then, taking the sum over all finite subsets of l, and noting that $\mathcal{B}$,, the left annihilator of A, can be re-interpreted as a giant l-columned matrix:
- the dual statement simplifies to $\mathcal{B} \overline{v} = \mathcal{O}$ with $\overline{v}$ restricted to taking on values which are 0 almost everywhere.

Axiomatization of 'Dual (A)'

For all column finite ${}_I A^J$ over R, and $\mathcal{B}$ the left annihilator subspace of A, interpreted as an $L \times I$ matrix for some index set L,

$$\mathcal{B}\overline{v} = 0 \implies \exists \overline{w}\, A\overline{w} = \overline{v}$$

where the vectors of variables have shapes ${}_{(I)}\overline{v}^1$ and ${}_{(J)}\overline{w}^1$.

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where the vectors of variables have shapes ${}_{(I)}\overline{v}^1$ and ${}_{(J)}\overline{w}^1$.

If A were row-finite and $\overline{v}$, $\overline{w}$, unrestricted, this would just be injectivity.

Axiomatization of ‘Dual (A)’ [Refresh: November 24]

The previous slide neglected to mention that ‘Dual (A)’ is a property of a particular left R-module E: it said nothing about where the truth of the formulas displayed is evaluated.

‘Dual (A)’, revised

A left R-module E satisfies ‘Dual (A)’ iff for all column finite ${}_IA^J$ over R, and $\mathcal{B}$ the left annihilator subspace of A, interpreted as an $L \times I$ matrix for some index set L, and for all finite $I' \ll I$ and all ${}_{I'}\overline{b}^1 \in E$,

$$\mathcal{B}^{I'}\overline{b} = 0 \implies E \models \exists \overline{w}\, A\overline{w} = \overline{b}$$

where the existential quantifier is taken in the weak sense: the solution $\overline{w}$ must have finite support.

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where the existential quantifier is taken in the weak sense: the solution $\overline{w}$ must have finite support.

This looks almost like the condition for injectivity, but quite skewed:

Review of injectivity

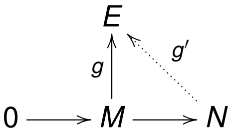
Characterization of injectivity

E is injective iff for all row finite matrices ${}_I A^J$ and ${}_L \mathcal{B}^I$ the left annihilator matrix of A ($\mathcal{B}$ is also necessarily row finite), and all ${}_I \overline{b}^1 \in E$,

$$\mathcal{B}\overline{b} = 0 \implies E \models \exists \overline{w} A\overline{w} = \overline{b},$$

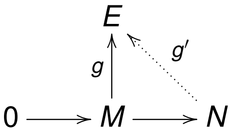
where the existential quantifier is taken in the strong sense: the solution $\overline{w}$ is in ${}_J E^1$.

Review of injectivity: construction of the homomorphism



Let $\overline{a}$ enumerate N, let $A\overline{x} = \overline{b}$ be the system of all linear equations with constants in M satisfied by $\overline{a}$.

Review of injectivity: construction of the homomorphism



Let $\overline{a}$ enumerate N, let $A\overline{x} = \overline{b}$ be the system of all linear equations with constants in M satisfied by $\overline{a}$.

Since this system is by definition satisfiable, it is formally consistent; and therefore so is its homomorphic image $A\overline{x} = g(\overline{b})$.

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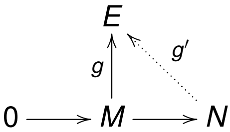

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So this system has a solution $\overline{b}'$ in E,

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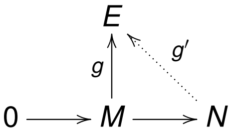

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Since this system is by definition satisfiable, it is formally consistent; and therefore so is its homomorphic image $A\overline{x} = g(\overline{b})$.

So this system has a solution $\overline{b}'$ in E,
and the map $\overline{b} \mapsto \overline{b}'$ is clearly a homomorphism g' extending g.

How did we get the axiomatization of Property A?

…by a lot of

simplification
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…

How did we get the axiomatization of Property A?

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. . .

So we probably have to re-entangle and complexify “Dual (A)” to interpret it algebraically.

Expanding Dual (A)

'Dual (A)', revised

A left R-module E satisfies 'Dual (A)' iff for all column finite ${}_I A^J$ over R, and $\mathcal{B}$ the left annihilator subspace of A, interpreted as an $L \times I$ matrix for some index set L, and for all finite $I' \ll I$ and all ${}_{I'}\overline{b}^1 \in E$,

$$\mathcal{B}^{I'}\overline{b} = 0 \implies E \models \exists \overline{w}\, A\overline{w} = \overline{b}$$

where the existential quantifier is taken in the weak sense: the solution $\overline{w}$ must have finite support.

Deconstruct the parts; decorate everything with their index sets:

Expanding Dual (A), continued

'Dual (A)', deconstructed

A left R-module E satisfies 'Dual (A)' iff

for all column finite ${}_I A^J$ over R,
and for $\mathcal{B}$ the left annihilator subspace of A, (interpreted as an $L \times I$ matrix for some index set L):

$$\text{for all finite } I' \ll I \text{ and all } {}_{I'}\overline{b}^1 \in E\,,$$

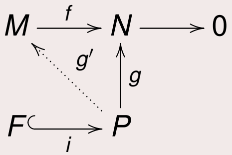
$${}_L\mathcal{B}^{I'}\,{}_{I'}\overline{b}^1 = {}_L0^1 \text{ implies that:}$$

$$\text{there is finite } J' \ll J \text{ and } {}_{J'}\overline{c}^1 \in E$$
$$\text{such that } {}_{I'}A^{J'}\,{}_{J'}\overline{c}^1 = {}_{I'}\overline{b}^1\,.$$

and then since $\mathcal{B}$ is column finite, we can then restrict to a finite subset $L' \ll L$.

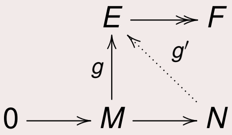
Categorical dual???

locally projective



F finitely generated.

dually locally projective



$E\,/\,F$ finitely generated.

Expanding Dual (A), continued

So what really is the relationship of the finite matrix ${}_{L'}\mathcal{B}^{I'}$ to A?

Expanding Dual (A), continued

So what really is the relationship of the finite matrix ${}_{L'}\mathcal{B}^{I'}$ to A?

Or do we need to ‘deconstruct’ A?

What role does A play in the characterization, as a possibly-infinite-in-both-dimensions column-finite matrix?