The Power Matrix, Coadjoint Action and Quadratic Differentials

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Abstract

The coefficients of a quadratic differential which is changing under the Loewner flow satisfy a well-known differential system studied by Schiffer, Schaeffer and Spencer, and others. By work of Roth, this differential system can be interpreted as Hamilton's equations. We apply the power matrix to interpret this differential system in terms of the coadjoint action of the matrix group on the dual of its Lie algebra. As an application, we derive a set of integral invariants of Hamilton's equations which is in a certain sense complete. In function theoretic terms these are expressions in the coefficients of the quadratic differential and Loewner map which are independent of the parameter in the Loewner flow.

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1 Introduction

1.1 Statement of results

Let $f(z)$ be a holomorphic function such that $f(0) = 0$ and $f'(0) \neq 0$. The "power matrix" of a locally conformal map f is the upper triangular matrix whose m,nth entry is the nth coefficient of the power series of f^m. This matrix representation has existed for quite some time in one form or another, and is closely related to the "Faá di Bruno formula" for the power series expansion of a composition of two functions (see [1]). In different contexts it has been observed to simplify complicated expressions or reveal identities in function theory. Jabotinski [3] showed that the Faber polynomials (or rather their derivatives) have a nice interpretation in terms of the power matrix, and went on to show that various identities relating to the Grunsky matrix and Faber polynomials have simple explanations in the matrix model. Identities concerning the coefficients of the Loewner flow in terms of the power matrix appear in Schiffer and Tammi [12], along with further applications to the Nehari and Grunsky inequalities.

The object of this paper is to exhibit more of this algebraic structure which has not yet been observed. A secondary purpose is to develop the theory of the power matrix in a systematic way, paying full attention to geometric and algebraic considerations such as the standard trick of trivializing the tangent and contangent bundle in terms of the Lie algebra. The trivialization is a key element in the results of this paper.

The main results are as follows:

1) Fix a quadratic differential $\zeta^{-2}P(\zeta)d\zeta^2$ and Loewner chain which is admissible for $\zeta^{-2}P(\zeta)d\zeta^2$, and pull it back to the disc to get a time-dependent quadratic differential $\zeta^{-2}Q_t(\zeta)d\zeta^2$. (This situation arises by combining the Loewner method and Schiffer's theorem and was investigated by Schaeffer and Spencer [10] and Schiffer [11] among others.) The time dependence of the coefficients of the quadratic differential are governed by a simple differential system ([10] p135). It is shown in the present paper that this differential system, when written in the matrix model, has a simple expression in terms of the coadjoint action of the power matrix on its Lie algebra (Theorem 2.)

Roth [9] showed that the quadratic differential is closely related to a Hamiltonian function arising in the control-theoretic view of Schiffer's equation, and that the differential system for the coefficients can be thought of as Hamilton's equations. This facilitated the recognition in the present work of the role of the coadjoint map (see discussion below). Roth also integrated this differential system. An alternate but equivalent form for the solution of this differential system in terms of the adjoint map is given here. This leads to the second main result.

2) We define an infinite series of functions on the cotangent bundle to the power matrix which are invariant under Hamilton's equations (Theorem 4), using the algebraic and geometric structure described above. In function theoretic terms, these are explicit expressions in the coefficients of the Loewner map and the quadratic differential which are time-independent (Corollary 2). These identities have two very interesting properties: they do not involve the infinitesimal generator in the Loewner equation, and they are independent of the functional under consideration. In some sense the identities are a complete set of invariants for Hamilton's equations, at least for finite coefficient estimates.

Although in principle this algebraic structure could have been observed from the work of Schaeffer and Spencer, it must be emphasized that it is work of Roth [9] that really makes this possible. (The fact that the relation to the coadjoint action has not been observed in over fifty years is evidence for this assertion). The reason is simply stated. Roth's remarkable result is that Schiffer's differential equation is equivalent to Pontryagin's maximum principle (in the context of the Loewner method). One of the key steps in demonstrating this equivalence was his observation that the coefficients of the quadratic differential and the coefficients of a "dual vector" arising in the control-theoretic picture satisfy the same differential system. When combined with the matrix model, this association of the coefficients of the quadratic differential with the dual vector leads directly to the expression of the differential system in terms of the coadjoint action given here.

This matrix model is natural from several points of view. Loewner's method is at heart a Lie-theoretic one; in fact it is much closer to Lie's conception in terms of transformation groups. The power matrix simply serves as a bridge between Loewner's description of locally biholomorphic functions as a semigroup of transformations and the currently fashionable idea of a Lie group. It should also be pointed out that the adjoint map is essential to the problem of constructing geometric invariants.

The problem of linking the semigroup perspective with the standard approach to Lie groups in different specific circumstances can be a difficult problem. One technique for doing so, the "method of prolongations" [6], produces exactly the picture derived here when applied to the group of two by two conformal matrices. We will not pursue this here, since the picture we obtain here does not require this general machinery.

The expression for Hamilton's equations given here (4.2) is standard in the theory of Hamiltonian systems on Lie groups ([4] chapter 12) and follows directly from our choice of Hamiltonian function. In the present case we are dealing with a non-abelian group so a term involving the adjoint map appears. The Lie group formalism is essential, since this crucial term disappears in the familiar form of Hamilton's equations on $\mathbb{R}^n$.

1.2 Outline

In Section 2 we describe the matrix group and develop its algebra and geometry. First the definitions are given in Section 2.1. The Lie algebra is described in Section 2.2. The tangent bundle, cotangent bundle and complex structure are defined in Section 2.3. In Section 2.4 we present some identities related to left and right multiplication on the tangent space level. These identities are the main tools for recognizing the matrix structure present in various function-theoretic objects such as the Loewner equation, the coefficients of the quadratic differential, and Schiffer's equation. In Section 3 we give the form of the Loewner equation in the matrix model.

The main results are presented in Section 4. In Section 4.1 we define the adjoint map and its relation to Hamilton's equations in our setting. Section 4.2 presents the relation between the coadjoint map and the differential system for the quadratic differential. The relation to Roth's work is clarified in Section 4.3, where his solution of the differential system is written in terms of the adjoint map.

Finally in Section 5 we define the integral invariants of Hamilton's equations. Explicit expressions are given in Section 5.2.

A notation key is provided in Section 6.

2 Algebraic and geometric structure of the group of locally conformal maps

In this section we define the algebraic and geometric structure of the matrix model, including the group itself and its Lie algebra, left and right multiplication, and the trivialization of the tangent and cotangent bundle through right multiplication.

2.1 The power matrix

It is possible to represent a function which is biholomorphic in a neighbourhood of 0 and normalized so that $f(0) = 0$ as an infinite matrix $[f]$. In this representation, composition of functions becomes matrix multiplication: $[f \circ g] = [f][g]$. Thus the matrix representation respects the semigroup structure.

Let $f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$. Then $[f]$ is the matrix whose entry $[f]^m_n$ in the mth row and nth column is the nth coefficient of the function $f(z)^m$. Here m and n range from 1 to ∞. Of course $[f]^1_n = f_n$; also $[f]^m_n$ are functions of $[f]^1_n$ for $m > 1$. The matrix is clearly upper triangular; thus matrix multiplication involves no infinite sums.

We show that $[f \circ g] = [f] \cdot [g]$. Let $g(z) = g_1 z + g_2 z^2 + \cdots$ be another such function, biholomorphic in a neighbourhood of 0. Then

$$\begin{aligned}(f \circ g)^m(z) &= \sum_{k=m}^{\infty} [f]^m_k \, g(z)^k \\ &= \sum_{k=m}^{\infty} [f]^m_k \sum_{n=k}^{\infty} [g]^k_n \, z^n \\ &= \sum_{n=m}^{\infty} (\sum_{k=m}^{n} [f]^m_k \, [g]^k_n) z^n\end{aligned}$$

and thus $[f\circ g]^m_n = \sum_{k=1}^m [f]^m_k\,[g]^k_n$. It immediately follows that if f^{-1} is the inverse of f in a neighbourhood of the origin, then $[f^{-1}] = [f]^{-1}$.

Thus we define $\mathbf{G}$ to be the group of upper triangular matrices arising from such maps.

Remark 1 (Negative powers of f). This matrix can be extended in two different ways. Jabotinski [3] includes the negative powers of f; i.e. he associates to f a doubly-infinite matrix with rows $m = -\infty, \ldots, \infty$ where for m negative the mth row consists of the coefficients of $1/f^{|m|}$. Schiffer and Tammi [12] include the coefficients of $\log(f(z)/z)$ as a zeroth row, along with the positive powers of the matrix. Both arise in applications involving Grunsky-type inequalities.

It's easy to see that the coefficients of powers of a locally univalent function g such that $g(0) = 0$ are polynomial functions in the coefficients of g. These polynomial functions are sometimes referred to as 'Bell polynomials'. There is a convenient rule for generating the entries of $[g]$ inductively.

Proposition 1.

$$[g]^{k+1}_n = \frac{k+1}{n}\sum_{l=1}^{n-k} l[g]^1_l\,[g]^k_{n-l}.$$

Proof. By definition

$$g(z)^{k+1} = \sum_{n=k+1}^{\infty} [g]^{k+1}_n z^n.$$

Differentiating both sides, we get

$$(k+1)g(z)^k \cdot g'(z) = \sum_{n=k+1}^{\infty} n\,[g]^{k+1}_n z^{n-1}.$$

Expanding the left side,

$$\begin{aligned}(k+1)g(z)^k \cdot g'(z) &= (k+1)\left(\sum_{m=k}^{\infty}[g]^k_m z^m\right)\left(\sum_{l=1}^{\infty} l\,[g]^1_l\,z^{l-1}\right)\\ &= (k+1)\sum_{m=k}^{\infty}\left(\sum_{l=1}^{m-k+1} l\,[g]^1_l\,[g]^k_{m-l+1}\right) z^m\end{aligned}$$

Letting $m = n-1$ in the last expression and equating coefficients proves the proposition. □

This gives us a completely algebraic description of the semigroup of locally univalent functions defined in a neighbourhood of the origin and such that $f(0) = 0$. It can be represented as the group $\mathbf{G}$ of infinite upper triangular matrices

$$\begin{pmatrix} x^1_1 & x^1_2 & x^1_3 & \cdots \\ 0 & x^2_2 & x^2_3 & \cdots \\ 0 & 0 & x^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $x^1_1 \neq 0$, the first row satisfies

$$\limsup |x^1_n|^{1/n} < \infty$$

and the remaining rows satisfy the relations given in Proposition 1.

2.2 Description of the Lie algebra

We now construct the Lie algebra of $\mathbf{G}$. Consider the tangent space at the identity of this matrix group, using the variation

$$G_\lambda(z) = z + \lambda h(z) + O(\lambda^2), \tag{2.1}$$

where λ is a complex parameter, and $h(z)$ is a holomorphic function in a neighbourhood of 0. We assume that $G_\lambda(0) = 0$ and $h(0) = 0$. The term $O(\lambda^2)$ is understood to be uniform on relatively compact sets in the domain of definition of $G(z)$. We have by an easy computation

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda]_n^m = m[h]_{n-m+1}^1.$$

So it is reasonable to make the following definition.

Definition 1. *Let $\mathfrak{g}$ denote the set of infinite upper triangular matrices whose entry in row m and column n is mh_{n-m+1} for some holomorphic function h defined in a neighbourhood of the origin. These matrices will be denoted by $\langle h \rangle$. i.e. if $h(z) = h_1 z + h_2 z^2 + \cdots$, then*

$$\langle h \rangle_n^m = mh_{n-m+1}.$$

Explicitly,

$$\langle h \rangle = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots \\ 0 & 2h_1 & 2h_2 & \cdots \\ 0 & 0 & 3h_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 2. Note that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda]_n^m \neq \left[\left.\frac{d}{d\lambda}\right|_{\lambda=0} G_\lambda \right]_n^m$$

in general; but rather

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda]_n^m = \left[\left.\frac{d}{d\lambda}\right|_{\lambda=0} (G_\lambda^m) \right]_n^1.$$

Remark 3. Note that we have extended the notation $[f]_n^1$ and applied it to functions are not elements of $\mathbf{G}$. For example, in the paragraph preceding Definition 1, h might not be a biholomorphism in a neighbourhood of 0; in Remark 2, G_λ^m certainly is not. Thus neither should be regarded as elements of $\mathbf{G}$. When applied to functions that are not in $\mathbf{G}$, the notation $[\,]_n^1$ will be simply taken to mean the nth coefficient of the function.

The Lie bracket of a pair of elements $\langle h \rangle, \langle j \rangle \in \mathfrak{g}$ is the commutator

$$[\langle h \rangle, \langle j \rangle] \equiv \langle h \rangle \langle j \rangle - \langle j \rangle \langle h \rangle \tag{2.2}$$

where $\langle h \rangle \langle j \rangle$ refers to the product of the two matrices.

Next we construct a useful basis, which is natural both algebraically and complex analytically. Define $\mathbf{e}_k = \left\langle z^{k+1} \right\rangle$; i.e. $\mathbf{e}_k$ is the matrix with n in the nth row and $n+k$th column for each n, and all other entries zero. For example,

$$\mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 3 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

We have

Proposition 2. *By the definition of the Lie bracket (2.2),*

$$[\mathbf{e}_k, \mathbf{e}_l] = (k-l)\mathbf{e}_{k+l}.$$

Proof. We have that $\mathbf{e}_k \cdot \mathbf{e}_l$ is the matrix with $n(n+k)$ in the nth row and $n+k+l$th column, and is otherwise zero. Similarly for $\mathbf{e}_l \cdot \mathbf{e}_k$ with the obvious changes. Thus, in the nth row and $n+k+l$th column of $[\mathbf{e}_k, \mathbf{e}_l]$ the entry is $n(n+k) - n(n+l) = (k-l)n$; all other entries are zero. □

Remark 4. Thus $\mathfrak{g}$ is the "positive part" of the Virasoro algebra with zero central charge [5]. (The author is grateful to D. Radnell for this observation.)

2.3 Tangent and cotangent bundle

The structure of the tangent and cotangent bundle, denoted $\mathrm{T}\mathbf{G}$ and $\mathrm{T}^*\mathbf{G}$ respectively, will now be described. We will not carefully develop the differentiable structure of $\mathbf{G}$, $\mathrm{T}\mathbf{G}$ and $\mathrm{T}^*\mathbf{G}$, so in a sense this description is purely formal. Where these issues threaten, function theory takes care of the problem in a standard way, at least for the results of this paper. This will of course be pointed out in the appropriate places.

An element of the tangent bundle $\mathrm{T}\mathbf{G}$ based at a point $[w_{t_0}]$ can be thought of as the derivative of a curve $[w_t] \in \mathbf{G}$ at $t = t_0$, that is

$$\left.\frac{d}{dt}\right|_{t=t_0} [w_t] \in \mathrm{T}_{[w_{t_0}]}\mathbf{G}.$$

where it is assumed that the curve is given by a variation of the form (2.1). Note that this new matrix is *not* an element of $\mathbf{G}$ and does not in general satisfy the relations of Proposition 1. We will denote the tangent space at $[w_{t_0}]$ by $\mathrm{T}_{[w_{t_0}]}\mathbf{G}$.

An element of the cotangent bundle $\mathrm{T}^*\mathbf{G}$ at a base point $[w_{t_0}]$ is a linear functional on $T_{[w_{t_0}]}\mathbf{G}$. Denote the cotangent space at $[w_{t_0}]$ by $\mathrm{T}^*_{[w_{t_0}]}\mathbf{G}$. Although $\mathbf{G}$ is infinite-dimensional, we will avoid possible problems by restricting our attention to linear functionals which are in some sense finite. We will make this precise shortly. For now the cotangent bundle will be treated formally.

We now define the right and left multiplication maps. On the group level, they are exactly what their names imply: for $[f], [g] \in \mathbf{G}$, the right multiplication map $\mathrm{R}[f]$ is $(\mathrm{R}[f])[g] = [g][f]$ and the left multiplication map $\mathrm{L}[f]$ is $(\mathrm{L}[f])[g] = [f][g]$. The derivatives of these maps act on $\mathrm{T}\mathbf{G}$ as follows.

Right multiplication is given by

$$\begin{aligned} \mathrm{R}[f]_* : \mathrm{T}_{[w_{t_0}]}\mathbf{G} &\to \mathrm{T}_{[w_{t_0}][f]}\mathbf{G} \\ \left.\frac{d}{dt}\right|_{t=t_0} [w_t] &\mapsto \left.\frac{d}{dt}\right|_{t=t_0} (\mathrm{R}[f]_*[w_t]) = \left(\left.\frac{d}{dt}\right|_{t=t_0} [w_t] \right) [f]. \end{aligned} \tag{2.3}$$

and similarly

$$\begin{aligned} \mathrm{L}[f]_* : \mathrm{T}_{[w_{t_0}]}\mathbf{G} &\to \mathrm{T}_{[f][w_{t_0}]}\mathbf{G} \\ \left.\frac{d}{dt}\right|_{t=t_0} [w_t] &\mapsto \left.\frac{d}{dt}\right|_{t=t_0} (L[f]_*[w_t]) = [f] \left(\left.\frac{d}{dt}\right|_{t=t_0} [w_t] \right). \end{aligned} \tag{2.4}$$

Thus by the linearity of matrix multiplication, $\mathrm{R}[f]_*$ and $\mathrm{L}[f]_*$ are also given by multiplication by $[f]$ on the right and left respectively. However we will retain the lower-star notation in order to distinguish between the maps on the group level and on the tangent space level.

Finally, we define the maps $R[f]^*$ and $L[f]^*$ on the cotangent bundle as follows. Given $\xi \in \mathrm{T}^*_{[w]}\mathbf{G}$, $R[f]^*\xi$ is the element of $\mathrm{T}^*_{[w][f]^{-1}}\mathbf{G}$ given by

$$(R[f]^*\xi)(v) = \xi(R[f]_* v)$$

for any $v \in \mathrm{T}_{[w][f]^{-1}}\mathbf{G}$. Similarly $L[f]^* : \mathrm{T}^*_{[w]}\mathbf{G} \to \mathrm{T}^*_{[f]^{-1}}\mathbf{G}$ is given by

$$(L[f]^*\xi)(v) = \xi(L[f]_* v)$$

for any $v \in \mathrm{T}_{[f]^{-1}[w]}\mathbf{G}$.

The tangent and cotangent bundle of any group are "trivializable": they can be written as a product of $\mathbf{G}$ with $\mathfrak{g}$ or $\mathfrak{g}^*$ respectively. Given $\frac{d}{dt}\big|_{t=t_0}[w_t] \in \mathrm{T}_{[w_{t_0}]}\mathbf{G}$ as above, we represent it with an element of $\mathbf{G} \times \mathfrak{g}$ as follows:

$$\begin{array}{rcl} \mathrm{T}\mathbf{G} & \cong & \mathbf{G} \times \mathfrak{g} \\ \left.\dfrac{d}{dt}\right|_{t=t_0} [w_t] & \mapsto & \left([w_{t_0}], \mathrm{R}[w_{t_0}^{-1}]_* \dfrac{d}{dt}[w_{t_0}] \right) \\ & = & ([w_{t_0}], \langle h \rangle), \end{array} \tag{2.5}$$

where $\langle h \rangle = \frac{d}{dt}\big|_{t=t_0}[w_t][w_{t_0}]^{-1} \in \mathfrak{g}$ according to the standard identification of $\mathfrak{g}$ with the tangent space at the identity $\mathrm{T}_{[z]}\mathbf{G}$ (see Definition 1). Similarly, given an element $\xi_{[w]} \in \mathrm{T}^*_{[w]}\mathbf{G}$, we represent it with an element of $\mathbf{G} \times \mathfrak{g}^*$:

$$\begin{array}{rcl} \mathrm{T}^*\mathbf{G} & \cong & \mathbf{G} \times \mathfrak{g}^* \\ \xi_{[w]} & \mapsto & ([w], \mathrm{R}[w]^*\xi_{[w]}) \\ & = & ([w], \alpha) \end{array} \tag{2.6}$$

where $\alpha \in T^*_{[z]}\mathbf{G} = \mathfrak{g}^*$.

The Lie algebra $\mathfrak{g}$ has an almost complex structure J given by multiplication by i; that is $J\langle h \rangle = \langle ih \rangle$. This almost complex structure is invariant under the map $\langle h \rangle \mapsto [w]\langle h \rangle [w]^{-1}$ for any $[w]$. To see this, simply note that the action of J on $\mathfrak{g}$ can be represented with multiplication by $i\mathrm{I}$ where I is the identity matrix:

$$[w]\mathrm{J}\langle h \rangle [w]^{-1} = [w]\, i\mathrm{I}\ \langle h \rangle [w]^{-1} = i\mathrm{I}\,[w]\langle h \rangle [w]^{-1} = \mathrm{J}[w]\langle h \rangle [w]^{-1}.$$

The almost complex structure on $\mathfrak{g}$ extends to $\mathrm{T}\mathbf{G}$ via the trivialization by right multiplication:

$$\mathrm{J}\langle h \rangle [w] = \langle ih \rangle [w].$$

It follows immediately from the definition that this almost complex structure is right invariant; since it is invariant under $\langle h \rangle \mapsto [w]\langle h \rangle [w]^{-1}$ for any $[w]$, it is also left invariant. That is

$$\mathrm{J}\,\mathrm{R}[w]_* = \mathrm{R}[w]_*\,\mathrm{J} \quad \text{and} \quad \mathrm{J}\,\mathrm{L}[w]_* = \mathrm{L}[w]_*\,\mathrm{J}.$$

In the trivialization, the almost complex structure has the form $\mathrm{J}([w], \langle h \rangle) = ([w], \langle ih \rangle)$.

The Lie algebra $\mathfrak{g}$ can be treated either as a real linear space with basis $\{\mathbf{e}_0, i\mathbf{e}_0, \mathbf{e}_1, i\mathbf{e}_1, \ldots\}$ or a complex linear space with basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots\}$. Strictly speaking the dual of the Lie algebra

$\mathfrak{g}^*$ should be regarded as the space of real linear functionals on $\mathfrak{g}$, but we can identify it with the space of complex linear functionals as follows. A real linear functional $\alpha_{\mathbb{R}}$ extends to a complex linear functional $\alpha_{\mathbb{C}}$ via

$$\alpha_{\mathbb{C}}(\langle h \rangle) = \alpha_{\mathbb{R}}(\langle h \rangle) - i\alpha_{\mathbb{R}}(\mathrm{J}\,\langle h \rangle).$$

Conversely, given a complex linear functional $\alpha_{\mathbb{C}}$ we can recover the real linear functional simply by taking the real part:

$$\alpha_{\mathbb{R}}(\langle h \rangle) = \mathrm{Re}(\alpha_{\mathbb{C}}(\langle h \rangle)).$$

It's not hard to check that under this identification $\mathrm{J}\alpha_{\mathbb{R}} \leftrightarrow i\alpha_{\mathbb{C}}$.

The natural complex structure on $\mathfrak{g}^*$ is given by $\mathrm{J}\alpha_{\mathbb{R}}(\langle h \rangle) \equiv \alpha_{\mathbb{R}}(\mathrm{J}\,\langle h \rangle)$, so we see that this just becomes multiplication by i in the representation of $\mathfrak{g}^*$ by complex linear functionals. As before we can extend J to $\mathrm{T}^*_{[w]}\mathbf{G}$ for each $[w]$ by right multiplication, and in the trivialization this of course has the form $\mathrm{J}([w], \alpha) = ([w], \mathrm{J}\alpha)$.

In the following we will always use the complex model of $\mathfrak{g}^*$ unless stated otherwise.

We now return to the issue of the behaviour of the linear functionals on $\mathfrak{g}$. This is bound together with the problem of constructing a differentiable structure on $\mathbf{G}$, which can be done in different ways depending on the function-theoretic application one has in mind. It is an interesting question whether there is a canonical way of doing this which is suitable for most "reasonable" applications.

However in this paper the applications are to finite coefficient functionals on the class of univalent functions. As will be seen, the consequence of this choice of application is to restrict to elements of $\mathfrak{g}^*$ which are zero when applied to all but finitely many basis vectors $\mathbf{e}_k \in \mathfrak{g}$. In the trivialization of the cotangent bundle this means that we consider only elements $([w], \alpha)$ where

$$\alpha = \sum_{s=0}^{n} \alpha^s \mathbf{e}_s^*$$

for some finite n and $\mathbf{e}_s^*$ are defined by

$$\mathbf{e}_s^*(\mathbf{e}_l) = \begin{cases} 1 & \text{if } s = l \\ 0 & \text{if } s \neq l. \end{cases} \tag{2.7}$$

Thus it is possible to avoid this issue.

2.4 Some identities for left and right multiplication

We now derive some identities related to the maps $L[F]_*$ and $R[F]_*$. These identities are a crucial tool for identifying the matrix structure present in expressions arising in function theory.

Proposition 3 (right multiplication). *Let j and f be functions which are biholomorphic in a neighbourhood of 0, such that $j(0) = 0$ and $f(0) = 0$. Then,*

$$\left[f^{m-1} j \circ f\right]_n^1 = \sum_k \langle j \rangle_k^m \, [f]_n^k \, .$$

Proof. Consider any variation $G_\lambda \circ f = f(z) + \lambda j \circ f + O(\lambda^2)$ as in Section 2.2. Because the convergence is uniform in a neighbourhood of 0, we can differentiate the Taylor series term by term. We have

$$\begin{aligned} \left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda \circ f]_n^m &= \left[\left.\frac{d}{d\lambda}\right|_{\lambda=0} (G_\lambda \circ f)^m \right]_n^1 \\ &= m \left[f^{m-1} j \circ f \right]_n^1. \end{aligned}$$

On the other hand, one can exploit the linearity of the matrix model to get that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda \circ f]_n^m = \left.\frac{d}{d\lambda}\right|_{\lambda=0} ([G_\lambda])\,[f] = m \sum_k [j]_{k-m+1}^1 \,[f]_n^k.$$

Now apply Definition 1. □

The next identity corresponds to left multiplication.

Proposition 4 (left multiplication). *Let j and g be functions which are biholomorphic in a neighbourhood of 0, and such that $j(0) = 0$ and $g(0) = 0$. Then,*

$$m \left[g^{m-1}\, g'\, j \right]_n^1 = \sum_k [g]_k^m \,\langle j \rangle_n^k$$

where one sums over the index k.

Proof. Consider the variation $g \circ G_\lambda$ with $G_\lambda(z) = z + \lambda j(z) + O(\lambda^2)$ where $j(0) = 0$. We have, since $O(\lambda^2)$ is uniform in some neighbourhood of the origin,

$$\begin{aligned} \left.\frac{d}{d\lambda}\right|_{\lambda=0} [g \circ G_\lambda]_n^m &= \left[\left.\frac{d}{d\lambda}\right|_{\lambda=0} (g \circ G_\lambda)^m \right]_n^1 \\ &= \left. \left[m\,(g \circ G_\lambda)^{m-1}\, g' \circ G_\lambda j \right]_n^1 \right|_{\lambda=0} \\ &= \left[m\, g^{m-1}\, g'\, j \right]_n^1 \end{aligned}$$

On the other hand, using the linearity of matrix multiplication,

$$\begin{aligned} \left.\frac{d}{d\lambda}\right|_{\lambda=0} [g \circ G_\lambda]_n^m &= \sum_k [g]_k^m \left.\frac{d}{d\lambda}\right|_{\lambda=0} [G_\lambda]_n^k \\ &= \sum_k [g]_k^m \,\langle j \rangle_n^k \end{aligned}$$

by Definition 1. □

Finally we give a mixed version of these two identities.

Proposition 5 (left and right multiplication). *Let j, f, and g be as in the previous two propositions. Then*

$$\left[g' \circ f\, j \circ f \right]_n^1 = \sum_{l,k} [g]_l^1 \,\langle j \rangle_k^l \,[f]_n^k.$$

Proof. Using the fact that $O(\lambda^2)$ is uniform on compact sets, we have

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [g \circ G_\lambda \circ f]_n^1 = \left[g' \circ f \, j \circ f\right]_n^1 .$$

On the other hand one can use the linearity of matrix multiplication to get that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} [g \circ G_\lambda \circ f]_n^1 = \sum_{l,k} [g]_l^1 \langle j \rangle_k^l [f]_n^k .$$

□

3 The Loewner equations

We give a representation of the Loewner equations

$$\frac{d}{dt} w_t(z) = -w_t(z) p_t(w(z)) \tag{3.1}$$

and

$$\frac{d}{dt} F_t(z) = z F_t'(z) p_t(z) \tag{3.2}$$

in terms of the power matrix. Here F_t is a Loewner chain, $w_t = F_t^{-1} \circ F$, and $p_t \in \mathcal{P}$ where $\mathcal{P}$ is the class of holomorphic functions from the disc into the right half plane normalized so that $p(0) = 1$.

Variations of the matrix form of the Loewner equation appear several times in the literature. Schiffer and Tammi ([12], equation (27)) give the full differential equation of Proposition 7 for slit mappings in terms of $[F_t]_n^m$, but without the explicit identification of the matrix $\langle z p_t(z) \rangle$, which simplifies the notation somewhat. The first row of Proposition 7 is given in Roth [8] (equation II.96) for the full class $\mathcal{P}$, and with notation nearly identical to that used below. Friedland and Schiffer [2] give a similar formula, again for the first row of the matrix, for an equation related to the Loewner equation usually called the "Friedland-Schiffer" equation. They give a matrix form for the infinitesimal generator, but with somewhat different notation, and restricted to slit mappings (equation 3.10).

First we write the ordinary Loewner equation in terms of the power matrix.

Proposition 6. *The Loewner equation (3.1), when written in terms of the coefficients of w_t and p_t, is equivalent to the matrix equation*

$$\frac{d}{dt} [w_t] = - \langle z p_t \rangle [w_t] .$$

Proof. If the matrix equation above holds, then the first row is the Loewner equation, expressed in terms of the coefficients of the Taylor series of w_t. Conversely, if the Loewner equation (3.1) holds, then

$$\frac{d}{dt} (w_t^m) = m w_t^{m-1} \frac{dw_t}{dt} = -m w_t^m p_t \circ w_t .$$

So, applying Proposition 3 with $j(z) = z p_t(z)$, and Definition 1,

$$\begin{aligned} \frac{d}{dt} [w_t]_n^m &= -m \left[w_t^m p_t \circ w_t\right]_n^1 \\ &= - \langle z p_t \rangle_k^m [w_t]_n^k \end{aligned}$$

□

Next we give a representation of the partial differential equation in the matrix model.

Proposition 7. *The Loewner partial differential equation (3.2), when written in terms of the coefficients of F_t and p_t, is equivalent to the matrix differential equation*

$$\frac{d}{dt}[F_t] = [F_t] \langle zp_t \rangle .$$

Proof. Assuming that (3.2) holds, we have that

$$\frac{d}{dt}(F_t^m) = mF_t^{m-1}\frac{dF_t}{dt} = mF_t^{m-1}zF_t'p.$$

Using Proposition 4 with $j(z) = zp_t(z)$ and Definition 1,

$$\begin{aligned} \frac{d}{dt}[F_t]_n^m &= m\left[F_t^{m-1}zF_t'p_t\right]_n^1 \\ &= [F_t]_k^m \langle zp_t \rangle_n^k . \end{aligned}$$

Conversely, if the matrix equation is satisfied, one can apply Proposition 4 to the first row of the equation to derive Loewner's equation for the coefficients.

An alternate proof is to simply differentiate the expression

$$[F_t]\,[w_t] = I$$

which follows directly from the fact that $F_t(w_t(z)) = z$. □

We will sometimes refer to the matrix equation itself as Loewner's equation.

Remark 5. The matrix representation of the Loewner equation does not have much to do with Loewner chains. In fact *any* one parameter family of mappings satisfies the equations above, although if F_t is not a Loewner chain the infinitesimal generator p_t need not be in $\mathcal{P}$. With suitable assumptions on the one-parameter family guaranteeing differentiability of the coefficients (e.g. local uniform convergence in a neighbourhood of zero as $t \to t_0$), one gets the matrix differential equations of Propositions 6 and 7 simply by using the left or right trivialization and setting $\langle zp_t \rangle = \mathrm{R}[w_t]_*(d[w_t]/dt)$ or $\langle zp_t \rangle = \mathrm{L}[F_t]_*(d[F_t]/dt)$ respectively.

4 Quadratic differentials under the Loewner flow and the adjoint map

4.1 The Adjoint maps and Hamilton's equations

Any Lie group acts on itself by conjugation. This gives rise to the adjoint and coadjoint actions on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively.

Definition 2. *Each element of* $\mathbf{G}$ *gives rise to an automorphism of the group via*

$$\begin{aligned} Ad[f] : \mathbf{G} &\to \mathbf{G} \\ [g] &\mapsto Ad[f](g) = [f][g][f]^{-1}. \end{aligned}$$

The derivative of this map at the identity is an automorphism of the Lie algebra $\mathfrak{g}$*:*

$$\begin{aligned} Ad[f]_* : \mathfrak{g} &\to \mathfrak{g} \\ \langle j \rangle &\mapsto Ad[f]_*(\langle j \rangle) = [f]\,\langle j \rangle\,[f]^{-1}. \end{aligned}$$

Remark 6. Note that in the definition of Ad_* we are using the fact that the derivative of left and right multiplication is also given by left and right multiplication.

Definition 3. *Each element of the Lie algebra gives rise to an endomorphism of the Lie algebra as follows:*

$$\begin{aligned} ad\langle j\rangle : \mathfrak{g} &\to \mathfrak{g} \\ \langle h\rangle &\mapsto [\langle j\rangle, \langle h\rangle]. \end{aligned}$$

The dual of this map is defined for fixed $\alpha \in \mathfrak{g}^*$ by

$$ad\langle j\rangle^* \alpha(\langle h\rangle) = \alpha(ad\langle j\rangle \langle h\rangle).$$

The next proposition gives the relation between ad and Ad.

Proposition 8. *If*

$$\frac{d}{dt}[\phi_t] = \langle j_t\rangle [\phi_t]$$

then for any $\langle h\rangle \in \mathfrak{g}$

$$\frac{d}{dt}\left(Ad[\phi_t^{-1}]_* \langle h\rangle\right) = -Ad[\phi_t^{-1}]_*[\langle j_t\rangle, \langle h\rangle]$$

and for any $\alpha \in \mathfrak{g}^*$

$$\frac{d}{dt}\left(Ad[\phi_t^{-1}]^*\alpha\right) = -ad\langle j_t\rangle^* Ad[\phi_t^{-1}]^*\alpha.$$

Proof. By differentiating $[\phi_t^{-1}][\phi_t]$ we have that

$$\frac{d}{dt}[\phi_t^{-1}] = -[\phi_t^{-1}]\langle j_t\rangle.$$

The first equation follows by differentiating $[\phi_t^{-1}]\langle h\rangle[\phi_t]$. The second equation follows directly from the first. □

Remark 7. Note that we are not making any assumptions on $\langle j_t\rangle$; all that is necessary is that the coefficients be differentiable and the matrix equation $d[\phi_t]/dt = \langle j_t\rangle[\phi_t]$ hold. The second equation of Proposition 8 only involves finite sums. The third equation can involve infinite sums; so it should be temporarily regarded as a formal relation. However we will be imposing restrictions that remove this problem (Remark 9 ahead).

Proposition 9 (Expression for $\mathbf{ad}^*$ in the basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots\}$). *Let*

$$\langle zp\rangle = \left\langle \sum_{n=0}^{\infty} p_n z^n \right\rangle = \sum_{n=0}^{\infty} p_n \mathbf{e}_n \in \mathfrak{g}$$

and

$$\alpha = \sum_{s=0}^{\infty} \alpha^s \mathbf{e}_s^* \in \mathfrak{g}^*.$$

Then

$$ad\langle zp\rangle^* \alpha = \sum_{s=0}^{\infty}\left(\sum_{j=s}^{\infty} \alpha^j (j-2s) p_{j-s}\right)\mathbf{e}_s^*;$$

thus the coefficients satisfy the differential system

$$\frac{d\alpha^s}{dt} = \sum_{j=s}^{\infty} \alpha^j (j-2s) p_{j-s}. \tag{4.1}$$

Proof.

$$\begin{aligned}
\operatorname{ad}\langle zp\rangle^* \mathbf{e}_j^*(\mathbf{e}_s) &= \mathbf{e}_j^*(\operatorname{ad}\langle zp\rangle(\mathbf{e}_s)) \\
&= \mathbf{e}_j^*\left(\left[\sum_{n=0}^{\infty} p_n \mathbf{e}_n, \mathbf{e}_s\right]\right) \\
&= \mathbf{e}_j^*\left(\sum_{n=0}^{\infty} p_n (n-s)\mathbf{e}_{n+s}\right) \\
&= p_{j-s}(j-2s)
\end{aligned}$$

for $j \geq s$ by Proposition 2, and is zero otherwise. So

$$\operatorname{ad}\langle zp\rangle^* \alpha(\mathbf{e}_s) = \sum_{j=s}^{\infty} \alpha^j (j-2s) p_{j-s}.$$

Now we can just use the linearity of $\operatorname{ad}\langle zp\rangle^* \alpha$. □

The differential system (4.1) can be thought of as the covector or "momentum" component of Hamilton's equations. Hamilton's equations are a system of ordinary differential equations on the cotangent bundle, which depend on a choice of Hamiltonian function. The form that Hamilton's equations take in the present setting follows from the general theory of Hamiltonian systems on Lie groups ([4] chapter 12). We will not discuss the general theory here, but rather we will simply define Hamilton's equations for our setting and specific choice of Hamiltonian. The necessary results will be explicitly derived, with the exception of Pontryagin's maximum principle (Theorem 3) which is taken from [9].

It should be pointed out that "Hamilton's equations" is a bit of a misnomer, since we are not dealing with the classical situation of variational calculus in which we are free to vary over the entire space of possible paths. However, this minor infraction in terminology is quite convenient.

We now define the Hamiltonian. By a "control" we mean a time-dependent element of $\mathfrak{g}$, which determines the direction of flow in $\mathbf{G}$. In other words a control is a choice of infinitesimal parameter in the Loewner equation. In general a Hamiltonian is a function of the position, vector and control. We are concerned with a specific Hamiltonian:

Definition 4. *The complex Hamiltonian $H : T^*\mathbf{G} \times \mathfrak{g} \to \mathbb{C}$ is the map*

$$H(\xi_{[w]}, \langle j\rangle) = (R[w]^* \xi_{[w]})(\langle j\rangle)$$

where $\langle j\rangle$ is a control. We use the standard identification of $T^*_{[z]}\mathbf{G}$ with $\mathfrak{g}^*$.

In the trivialization $H : \mathbf{G} \times \mathfrak{g}^* \times \mathfrak{g}$ this has the form

$$H([w], \alpha, \langle j\rangle) = \alpha(\langle j\rangle).$$

Finally we let $H_{\mathbb{R}} = Re\, H$, so that

$$H_{\mathbb{R}}([w], \alpha_{\mathbb{R}}, \langle j\rangle) = \alpha_{\mathbb{R}}(\langle j\rangle) = Re\, \alpha(\langle j\rangle).$$

For this choice of Hamiltonian, Hamilton's equations take the following form.

Definition 5. *Hamilton's equations for the Hamiltonian in Definition 4 are as follows.*

$$\frac{d}{dt}[w_t] = \langle j_t \rangle [w_t] \tag{4.2}$$

$$\frac{d\alpha_t}{dt} = -ad \langle j_t \rangle^* \alpha_t. \tag{4.3}$$

Remark 8. Strictly speaking Hamilton's equations are given by taking the real part of the second set of equations; but by using the identification described at the end of Section 2.3 and the fact that the almost complex structure J commutes with left and right multiplication we can write them in the form above.

Remark 9 (Finiteness assumption). In the applications to follow, all but finitely many of the coefficients $\alpha^s = \alpha(\mathbf{e}_s)$ are zero, so this differential system is finite. This will always be assumed unless otherwise specified.

The position component of Hamilton's equations is just the Loewner equation. Given a solution to the Loewner equation, it is easy to find a solution to the "covector" component of Hamilton's equations. This is the content of the next theorem.

Theorem 1. *Assume that*

$$\frac{d}{dt}[\phi_t] = \langle j_t \rangle [\phi_t]$$

for $\langle j_t \rangle$ measurable in t, for almost all $t \in [0, T]$. The unique solution of

$$\frac{d\alpha_t}{dt} = -ad \langle j_t \rangle^* \alpha_t$$

with endpoint α_T is

$$\alpha_t = Ad[\phi_t^{-1}]^* Ad[\phi_T]^* \alpha_T.$$

Proof. α_t clearly satisfies the final condition. By Proposition 8 it satisfies the differential equation. Note that this is a finite system of ordinary differential equations by Remark 9. Uniqueness follows from the standard theory of ordinary differential equations. □

Remark 10. This solution of Hamilton's equations appears in [9], in a different form. This will be made explicit in Section 4.3.

4.2 Quadratic differentials and the coadjoint map

We now show that the flow of a quadratic differential under the Loewner equation is governed by the coadjoint map.

Let $\mathcal{S}^M$ denote the class of univalent functions from the disc into $\mathbb{D}^M = \{z : |z| < M\}$ satisfying the normalization $F(0) = 0$ and $F'(0) = 1$. The full class $\mathcal{S}$ of normalized univalent functions on $\mathbb{D}$ will be represented by the case $M = \infty$.

Let $\zeta^{-2}P^M(\zeta)d\zeta^2$ be a quadratic differential on $\mathbb{D}^M$ which satisfies $P^M(\zeta) \geq 0$ for $\zeta \in \partial\mathbb{D}$. A trajectory of $\zeta^{-2}P^M(\zeta)d\zeta^2$ is a curve $\gamma(t)$ such that $\gamma^{-2}P^M(\gamma)\dot{\gamma}^2 \leq 0$, where $\dot{\gamma}$ denotes the derivative of γ in t. A map $F : \mathbb{D} \to \mathbb{C}$ is said to be admissible for $\zeta^{-2}P^M(\zeta)d\zeta^2$ if it maps onto $\mathbb{D}^M$ minus a union of trajectories of $\zeta^{-2}P^M(\zeta)d\zeta^2$. In particular, $\mathbb{D}^M \backslash F(\mathbb{D})$ has measure zero.

We now define the Loewner system for a quadratic differential which arises from combining the Schiffer and Loewner equations ([9] equation (22), [10] p 135, [11]).

Definition 6 (Loewner system for a quadratic differential). *Let $\zeta^{-2}P_M(\zeta)d\zeta^2$ be a quadratic differential on $\mathbb{D}^M$ which satisfies $P_M(\zeta) \geq 0$ for $\zeta \in \partial\mathbb{D}^M$. Let $F \in \mathcal{S}^M$ be admissible for this quadratic differential, and F_t be a Loewner chain satisfying the normalization $F_t'(0) = e^t$ and $F_t(0) = 0$ such that $F_0 = F$ and $F_T(z) = Mz$ for $T = \log M$. Let $w_t : \mathbb{D} \to \mathbb{D}$ be the transition functions $w_t(z) = F_t^{-1} \circ F$, which satisfy the Loewner equation*

$$\frac{d}{dt}[w_t] = -\langle zp_t\rangle [w_t]$$

for $p_t \in \mathcal{P}$ measurable in t, for almost all $t \in [0, T]$.

We then define a one-parameter flow of quadratic differentials $\zeta^{-2}Q_t(\zeta)d\zeta^2$ as follows:

$$\frac{Q_t(\zeta)}{\zeta^2}d\zeta^2 = \frac{P_M(F_t(\zeta))}{F_t(\zeta)^2}F_t'(\zeta)^2 d\zeta^2.$$

In the case that $M = \infty$ we replace $\mathbb{D}^M$ with $\mathbb{C}$ and $\zeta^{-2}P_\infty(\zeta)d\zeta^2$ is a quadratic differential on $\mathbb{C}$.

This immediately implies the identity

$$\frac{Q_t(w_t(\zeta))}{w_t(\zeta)^2}w_t'(z)^2 d\zeta^2 = \frac{Q_0(\zeta)}{\zeta^2}d\zeta^2. \tag{4.4}$$

We then have the following theorem.

Theorem 2. *Consider the Loewner system of Definition 6 for $M \in [1, \infty]$. Let Q_t have the expansion*

$$Q_t(\zeta) = \sum_{s=-\infty}^{\infty} d_s(t)\zeta^{-s}$$

and $p_t(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then, forming the covector

$$\mathbf{d} = \sum_{s=1}^{\infty} d_s \mathbf{e}_s^*$$

we have

$$\frac{d\mathbf{d}}{dt} = -ad\,\langle zp_t\rangle^* \,\mathbf{d}.$$

Proof. The identity (4.4) implies that

$$\frac{\partial Q_t}{\partial t}(\zeta) = \zeta p_t(\zeta)\frac{\partial Q_t}{\partial \zeta}(\zeta) + 2\zeta p_t'(\zeta)Q_t(\zeta).$$

A computation shows that this identity implies that d_s satisfies the differential system 4.1 for $s \geq 1$ ([9] Lemma 4.4, [10] p 135, both in slightly different notation). Thus the theorem follows from Proposition 9. □

Remark 11. This theorem can be shown to hold without the finiteness assumption of Remark 9, although care needs to be taken in the interpretation of the expression ad $\langle zp\rangle^* \alpha$.

Remark 12 (On $d_0(t)$). The $s = 0$ coefficient of the quadratic differential does not satisfy the differential system 4.1. The particular value of this coefficient is related to the maximal Hamiltonian condition of Pontryagin's maximum principle [9]. Since the expressions for da_s/dt in Proposition 9 do not involve a_0, this does not affect the validity of Theorem 2.

Remark 13. It will be convenient to write the Loewner system Definition 6 in terms of the maps $v_t = e^t w_t$ as in [8]. In this case it is easily checked that v_t satisfies the Loewner equation

$$\frac{d}{dt}[v_t] = \langle j(p_t)\rangle [v_t] \tag{4.5}$$

with $\langle j(p_t)\rangle = \left\langle z(1 - p_t(e^{-t}z))\right\rangle$. The quadratic differential also takes another form. Let $\xi = e^t\zeta$ and $\hat{Q}_t(\xi) = Q_t(\zeta)$; i.e. $\xi^{-2}\hat{Q}(\xi)d\xi^2$ is a quadratic differential on $\mathbb{D}^{e^t}$. This implies that $\xi^{-2}\hat{Q}_t(\xi)d\xi^2 = \zeta^{-2}Q_t(\zeta)d\zeta^2$. The identity 4.4 is then equivalent to

$$\frac{\hat{Q}_t(v_t(\zeta))}{v_t(\zeta)^2} v_t'(\zeta)^2 d\zeta^2 = \frac{\hat{Q}_0(\zeta)}{\zeta^2} d\zeta^2.$$

Since $v_t = e^t w_t$, $\hat{Q}_0 = Q_0$. This system also has the advantage that $P_M = \hat{Q}_T$. The coefficients of $\hat{Q}$ are given by

$$\hat{Q}_t(\xi) = \sum_{s=-\infty}^{\infty} c_s(t)\xi^{-s},$$

where $c_s(t) = d_s(t)e^{-st}$. We then have, forming the covector

$$\mathbf{c} = \sum_{s=1}^{\infty} c_s \mathbf{e}_s^*$$

that

$$\frac{d\mathbf{c}}{dt} = -\mathrm{ad}\,\langle j_t\rangle^*\,\mathbf{c}.$$

This follows easily by simply observing that the coefficients of $j(p_t)$ are $e^{-st}p_s(t)$ for $s \geq 1$. Alternately one can repeat the proof of Theorem 2.

Remark 14. Note that Theorem 2 has nothing to do with a particular functional. If one does specify a functional, and P_M is determined by Schiffer's equation, this imposes endpoint conditions for the differential system 4.1, as will be seen in the next section.

4.3 Roth's dual system and Pontryagin's maximum principle

As mentioned in the introduction, both the differential system (4.1) and the power matrix have been known for quite some time, so Theorem 2 could have been observed long ago. However, a crucial insight was missing, which was provided in Roth [9]. In the course of demonstrating the equivalence of Pontryagin's maximum principle and Schiffer's equation in the context of the Loewner flow, a natural dual system was constructed, and it was shown that the coefficients of the quadratic differential and the coefficients of this dual system both satisfy (4.1). Thus, the coefficients of the quadratic differential can be identified with a covector, and with the Hamiltonian of Definition 4.

After this identification is made, and the formalism of the cotangent bundle to the power matrix group is developed, the role of the coadjoint map is easily recognized. In this section we will explain this last fact and along the way observe a few more identities.

Remark 15 (On terminology). There is a potentially confusing point of terminology: the term "adjoint" has (at least) two different meanings. First, it can refer to the action by conjugation on the group and Lie algebra as in Section 4.1. Second, the terms "adjoint vector" or "adjoint equation/system" can refer to dual vectors and the differential equation they satisfy, as in [9].

In order to avoid potential confusion, from here on "adjoint" will have only the first meaning, and we will use the term "dual system" and "dual vector" or "covector" in connection with the second meaning.

We begin by restating Roth's version of Pontryagin's maximum principle in terms of the power matrix. We will restrict our attention to coefficient functionals of finite order, say ReΦ where Φ has complex derivative $\Lambda_{[f]}$ ($\Lambda[f;\cdot]$ in the notation of Pommerenke [7]). Pontryagin's maximum principle can be roughly thought of as Hamilton's equations with a final condition (the 'tranversality condition') along with a maximality condition for the Hamiltonian.

Note that Roth's theorem is more general than the statement below, as we are restricting to finite functionals.

Theorem 3 (Pontryagin's maximum principle for $\mathcal{S}^M$ (Roth)). *Let $F \in \mathcal{S}^M$ be extremal for a finite coefficient functional $Re\,\Phi$, i.e.*

$$\max_{f\in\mathcal{S}^M} Re\,\Phi(f) = Re\,\Phi(F),$$

and Λ be the complex derivative of Φ. Let F_t be a Loewner chain with transition functions $w_t = F_t^{-1}\circ F$ and infinitesimal generator $p_t \in \mathcal{P}$ such that $e^T w_T(z) = F(z)$, $w_0(z) = z$, $w_t'(0) = e^t$ and $w_t(0) = 0$ for all $t \in [0,T]$ where $T = \log M$. Denote $v_t = e^t w_t$. Let $\alpha_t \in \mathfrak{g}^*$ be the solution of Hamilton's equations

$$\begin{aligned} \frac{d}{dt}[v_t] &= \langle j(p_t)\rangle\,[v_t] \\ \frac{d\alpha_t}{dt} &= -ad\,\langle j(p_t)\rangle^*\,\alpha_t \end{aligned}$$

satisfying the transversality condition

$$\alpha_T = -R[v_T]^* d\Phi_{[v_T]}.$$

Define the control function $\langle j(h)\rangle = \left\langle z(1-h(e^{-t}z))\right\rangle$ for $h \in \mathcal{P}$. Then for H as in Definition 4, we have

$$\max_{h\in\mathcal{P}} Re\,H\left([v_t],\alpha_t,-\langle j(h)\rangle\right) = Re\,H\left([v_t],\alpha_t,-\langle j(p_t)\rangle\right).$$

Proof. This is nothing more than Theorem 4.1 of [9] in different notation, so we need only make some translations. By Proposition 9, the unique solution of Hamilton's equations and the transversality condition is

$$\begin{aligned} \alpha_t &= -\mathrm{Ad}[v_t^{-1}]^*\mathrm{Ad}[v_T]^*\mathrm{R}[v_T]^* d\Phi_{[v_T]} \\ &= -\mathrm{Ad}[v_t^{-1}]^*\mathrm{L}[F]^* d\Phi_{[F]}. \end{aligned}$$

Since $d\Phi = \Lambda$ we then have for $\langle j\rangle \in \mathfrak{g}$,

$$\begin{aligned} \mathrm{H}\left([v_t],\alpha_t,-\langle j\rangle\right) &= -\mathrm{Ad}[v_t^{-1}]^*\mathrm{L}[F]^*\Lambda_{[F]}(-\langle j\rangle) \\ &= \Lambda_{[F]}\left([F][v_t^{-1}]\,\langle j\rangle\,[v_t]\right). \end{aligned}$$

Again applying Proposition 5 with $g = F\circ v_t^{-1}$, $j(z) = z(1-h(e^{-t}z))$ and $f(z) = v_t(z)$ we have

$$\mathrm{H}\left([v_t],\alpha_t,-\langle j\rangle\right) = \Lambda\left[F;\frac{F'(z)}{v_t'(z)}v_t(z)(1-h(e^{-t}v_t(z)))\right],$$

which is just expression (12) of [9] Theorem 4.1. □

Application of control theory requires the construction of a dual system to Loewner's equation. In Roth's dual system the covector is the linear functional given by

$$\langle j \rangle [v_t] \mapsto \Lambda_{[F]} \left[\frac{F'(z)}{v_t'(z)} j \circ v_t \right] \tag{4.6}$$

for $\langle j \rangle [v_t] \in \mathrm{T}_{[v_t]}\mathbf{G}$ (cf [9] p 404 and the proof of Theorem 4.1). This should be regarded as an element of $\mathrm{T}^*_{[v_t]}\mathbf{G}$. We use the trivialization by right multiplication $\mathrm{T}^*\mathbf{G} \cong \mathbf{G} \times \mathfrak{g}^*$ in order to give this covector a simple form. Applying Proposition 5 with $g = F_t = F \circ v_t^{-1}$, $j = j$ and $f = v_t$ we have that

$$\left[\frac{F'}{v_t'} j \circ v_t \right]_n^1 = \left[[F][v_t]^{-1} \langle j \rangle [v_t] \right]_n^1 .$$

Thus applying the covector to a vector $\langle j \rangle [v_t] \in \mathrm{T}_{[v_t]}\mathbf{G}$, we get

$$\begin{aligned} \Lambda_{[F]} \left[\frac{F'}{v_t'} j \circ v_t \right] &= \Lambda_{[F]} \left[[F][v_t]^{-1} \langle j \rangle [v_t] \right] \\ &= \Lambda_{[F]} \left[L[F]_* \mathrm{Ad}[v_t]_*^{-1} \langle j \rangle \right] \\ &= \mathrm{Ad}[v_t]^{-1^*} L[F]^* \Lambda_{[F]} \left[\langle j \rangle \right] \\ &= \mathrm{Ad}[v_t]^{-1^*} \mathrm{Ad}[v_T]^* R[F]^* \Lambda_{[F]} \left[\langle j \rangle \right] \end{aligned} \tag{4.7}$$

Setting

$$\alpha_T = -R[F]^* \Lambda_{[F]} \in \mathfrak{g} \tag{4.8}$$

and

$$\alpha_t = \mathrm{Ad}[v_t]^{-1^*} \mathrm{Ad}[v_T]^* \alpha_T \tag{4.9}$$

we see by Theorem 1 that α_t is a solution of the differential equation $d\alpha/dt = -\mathrm{ad} \langle j \rangle^* \alpha$. Thus we see that the solution to Hamilton's equations given in Theorem 1 is exactly the left hand side of (4.7). This is Roth's expression for the solution to the differential system (4.1).

The coefficients of the covector for $s \geq 1$ are given by

$$c_s(t) = -\alpha_t(\mathbf{e}_s) = -\mathrm{Ad}[v_t]^{-1} \mathrm{Ad}[v_T]^* \alpha_T(\mathbf{e}_s).$$

In Roth's notation, since $\mathbf{e}_s = \langle z^{s+1} \rangle$ and

$$\left[\frac{F'}{v_t'} v_t^{s+1} \right]_n^1 = \left[[F][v_t]^{-1} \langle z^{s+1} \rangle [v_t] \right]_n^1 ,$$

we have

$$c_s(t) = \Lambda_{[F]} \left[\frac{F'}{v_t'} v_t^{s+1} \right] . \tag{4.10}$$

and the functions $c_s(t)$ satisfy the differential system (4.1) with final condition $c_s(T) = \Lambda_{[F]}[F^{s+1}]$. Conversely, if the functions $c_s(t)$ satisfies the differential system (4.1) for $s \geq 1$ with this final condition, they must be given by $\alpha_t = \mathrm{Ad}[v_t]^{-1^*} \mathrm{Ad}[v_T]^* \alpha_T$ with $\alpha_T(\mathbf{e}_s) = -\Lambda_{[F]}[F^{s+1}] = -c_s(T)$.

We now examine the values of the covector at the endpoints $t = 0$ and $t = T$. Note that the coefficients of the positive and negative powers of ξ in $\hat{Q}_t$ are related by $c_{-s}(t) = \overline{c_s(t)}e^{-2st}$, since $\hat{Q}_t \geq 0$ on $\partial\mathbb{D}^{e^t}$. We have that, for $s \geq 1$,

$$\begin{aligned} c_s(0) &= \mathrm{Ad}[v_0]^{-1^*}\mathrm{Ad}[v_T]^*\alpha_T(\mathbf{e}_s) \\ &= \mathrm{L}[F]^*\Lambda_{[F]}(\mathbf{e}_s) \\ &= \Lambda_{[F]}([F]\left\langle z^{s+1}\right\rangle) \\ &= \Lambda_{[F]}(z^{s+1}F'(z)) \end{aligned} \tag{4.11}$$

where in the last step we have used Proposition 4 with $m = 1$, $j(z) = z^{s+1}$ and $g = F$. Similarly we have for $s \geq 1$

$$\begin{aligned} c_s(T) &= \mathrm{Ad}[v_T]^{-1^*}\mathrm{Ad}[v_T]^*\alpha_T(\mathbf{e}_s) \\ &= \mathrm{R}[F]^*\Lambda_{[F]}(\mathbf{e}_s) \\ &= \Lambda_{[F]}(\left\langle z^{s+1}\right\rangle [F]) \\ &= \Lambda_{[F]}(F(z)^{s+1}) \end{aligned} \tag{4.12}$$

where we have used Proposition 3 with $m = 1$, $j(z) = z^{s+1}$ and $f = F$. Of course both expressions agree with the left hand side of (4.7) with $j(z) = z^{s+1}$.

The values of the covector at the endpoints of the curve are the coefficients of the quadratic differential appearing in Schiffer's equation (with the exception of the zeroth coefficient; see Remark 12). We demonstrate this in the most familiar case $M = \infty$. (The general case can be seen by using a version of Schiffer's theorem for $\mathcal{S}^M$ given in [9] Theorem 4.7 and performing the same computations as below. One needs to take care with the notation: "P" in Theorem 4.7 [9] is denoted "Q_T" in the present paper).

Consider then the case $M = \infty$, i.e. the full class $\mathcal{S}$. We have that the P and Q appearing in Schiffer's equation (P_∞ and Q_0 in the notation here) satisfy

$$\frac{P_\infty(F(\zeta))}{F(\zeta)^2}F'(\zeta)^2 d\zeta^2 = \frac{Q_0(\zeta)}{\zeta^2}d\zeta^2$$

where P_∞ and Q_0 are given by

$$P_\infty(w) = \Lambda_{[F]}\left[\frac{F(z)^2}{w - F(z)}\right]$$

and

$$Q_0(\zeta) = \frac{1}{2}\Lambda_{[F]}\left[zF'(z)\frac{\zeta + z}{\zeta - z}\right] + \frac{1}{2}\Lambda_{[F]}\left[zF'(z)\frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z}\right] - \mathrm{Re}\Lambda_{[F]}\left[F(z)\right].$$

Expanding these in a power series in w and ζ we get the expressions (4.11) and (4.12) above.

5 Application: integrals of motion of Hamilton's equations

5.1 Definition and interpretation

It is interesting that the solution of Hamilton's equations given by Theorem 1 are independent of the control, and can be written entirely in terms of the point $[v_t] \in \mathbf{G}$ (or $[w_t]$ depending on the choice of control system) and the terminal covector. (Of course, the curve itself depends on the control). It is therefore easy to determine integrals of motion of Hamilton's equation from this solution. That is, there are certain expressions in the coefficients of α_t which are independent of t, depending only on the endpoint. The conserved quantity is easily written in terms of the adjoint map: since $\alpha_t = \mathrm{Ad}[v_t^{-1}]^*\mathrm{Ad}[v_T]^*\alpha_T$, the quantity

$$\mathrm{Ad}[v_t]^*\alpha_t = \mathrm{Ad}[v_T]^*\alpha_T$$

is constant. Thus we have the following theorem. (Note that the upper index α^s denotes the component of α and the lower index α_t denotes dependence of α on t.)

Theorem 4. *Let $[\phi_t]$ and α_t satisfy Hamilton's equations*

$$\begin{aligned} \frac{d}{dt}[\phi_t] &= \langle j_t\rangle [\phi_t] \\ \frac{d\alpha_t}{dt} &= -ad\,\langle j_t\rangle^* \alpha_t \end{aligned}$$

on some interval $[0,T]$ where T can be ∞, and such that for some integer $n > 1$ $\alpha(\mathbf{e}_s) = 0$ for $s \geq n$. Then setting $\alpha^s(t) = \alpha(\mathbf{e}_s)$ we have

$$I_m = -\sum_{s=0}^{n-1} \alpha^s(t)\mathbf{e}_s^*(Ad[\phi_t]_*\mathbf{e}_m)$$

are independent of t.

Proof. On any interval $[0,T']$, α_t is given by $\alpha_t = \mathrm{Ad}[\phi_t^{-1}]^*\mathrm{Ad}[\phi'_T]^*\alpha_{T'}$ by Theorem 1. Thus $\mathrm{Ad}[\phi_t]^*\alpha_t$ is constant. In particular $I_m = -\mathrm{Ad}[\phi_t]^*\alpha_t\,(\mathbf{e}_m)$ is constant for each m, which is the expression above. □

Remark 16. Note that $\mathbf{e}_s^*\,(\mathrm{Ad}[\phi_t]_*\mathbf{e}_m) = 0$ for $s < m$; thus the integrals I_m depend only on those $\alpha^s(t)$ with $s \geq m$.

In particular, we have the following Corollary.

Corollary 1. *If $[w_t]$ and Q_t are as in Definition 6, and $[v_t]$ and $\hat{Q}_t$ as in Remark 13, then the quantities*

$$I_m = \sum_{s=1}^{n-1} d_s(t)\mathbf{e}_s^*\,(Ad[w_t]_*\mathbf{e}_m)$$

are constant in t. In terms of $[v_t]$ and $\hat{Q}_t$ as in Remark 13, the quantities above have the alternate expression

$$I_m = \sum_{s=1}^{n-1} c_s(t)\mathbf{e}_s^*\,(Ad[v_t]_*\mathbf{e}_m)$$

are independent of t.

Proof. The fact that either expression for I_m is conserved follows from Theorems 2 and 4, and Remark 13. (The value of the zeroth coefficient does not enter into the expressions for $m \geq 1$ by Remark 16). That the two expressions are equal is just a matter of shuffling some factors of e^{-st}: it's not hard to check that $\mathrm{Ad}[v_t]_*\mathbf{e}_s = \mathrm{Ad}[e^t z]_*\mathrm{Ad}[w_t]_*\mathbf{e}_s$ and $\mathrm{Ad}[e^t z]_*\mathbf{e}_s = e^{-st}\mathbf{e}_s$, so $\mathbf{e}_s^*(\mathrm{Ad}[v_t]_*\mathbf{e}_m) = e^{-st}\mathbf{e}_s^*(\mathrm{Ad}[w_t]_*\mathbf{e}_m)$; then just observe that $d_s(t) = c_s(t)e^{-st}$. □

Remark 17. If P_M is determined from the complex derivative Λ of a functional by Schiffer's equation, then the constants I_m are determined by

$$I_m = \mathrm{L}[F]^* \Lambda_{[F]}[\mathbf{e}_m] = \Lambda_{[F]}[z^{s+1} F'(z)]$$

by the discussion following Theorem 3.

Remark 18. These invariants have two remarkable properties. First, they do not explicitly depend on the infinitesimal generator in the Loewner equation. Second, they are independent of the particular functional in question; thus if P_M is determined by Schiffer's equation from some functional, the expressions above are constant no matter what that original functional was. The particular functional only determines n and the constant values of I_m.

5.2 Explicit expressions for the integral invariants

We provide a table of the quantities $\mathbf{e}_s^*(\mathrm{Ad}[v_t]_* \mathbf{e}_m)$, which can be computed from the entries of the upper triangular matrix $\mathrm{Ad}[v_t]_* \mathbf{e}_m$. (Note that the quantities are simplified by the assumption that $a_1 = 1$, but the general case poses no additional difficulty).

$\underline{m = 0}$

$$\begin{aligned}
\mathbf{e}_0^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= 1 \\
\mathbf{e}_1^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= a_2 \\
\mathbf{e}_2^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= 2a_3 - 2a_2^2 \\
\mathbf{e}_3^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= 3a_4 - 8a_3a_2 + 5a_2^3 \\
\mathbf{e}_4^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= 4a_5 - 14a_4a_2 - 6a_3^2 + 30a_3a_2^2 - 14a_2^4 \\
\mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_0) &= 5a_6 - 22a_5a_2 - 18a_4a_3 + 56a_4a_2^2 + 49a_3^2a_2 - 112a_3a_2^2 + 42a_2^5
\end{aligned}$$

$\underline{m = 1}$

$$\begin{aligned}
\mathbf{e}_1^*(\mathrm{Ad}[v_t]_*\mathbf{e}_1) &= 1 \\
\mathbf{e}_2^*(\mathrm{Ad}[v_t]_*\mathbf{e}_1) &= 0 \\
\mathbf{e}_3^*(\mathrm{Ad}[v_t]_*\mathbf{e}_1) &= a_3 - a_2^2 \\
\mathbf{e}_4^*(\mathrm{Ad}[v_t]_*\mathbf{e}_1) &= 2a_4 - 6a_3a_2 + 4a_2^3 \\
\mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_1) &= 3a_5 - 12a_4a_2 - 5a_3^2 + 28a_3a_2^2 - 14a_2^4
\end{aligned}$$

$\underline{m = 2}$

$$\begin{aligned}
\mathbf{e}_2^*(\mathrm{Ad}[v_t]_*\mathbf{e}_2) &= 1 \\
\mathbf{e}_3^*(\mathrm{Ad}[v_t]_*\mathbf{e}_2) &= -a_2 \\
\mathbf{e}_4^*(\mathrm{Ad}[v_t]_*\mathbf{e}_2) &= a_2^2 \\
\mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_2) &= a_4 - 2a_3a_2
\end{aligned}$$

$\underline{m = 3}$

$$\begin{aligned}
\mathbf{e}_3^*(\mathrm{Ad}[v_t]_*\mathbf{e}_3) &= 1 \\
\mathbf{e}_4^*(\mathrm{Ad}[v_t]_*\mathbf{e}_3) &= -2a_2 \\
\mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_3) &= -a_3 + 4a_2^2
\end{aligned}$$

$\underline{m = 4}$

$$\begin{aligned} \mathbf{e}_4^*(\mathrm{Ad}[v_t]_*\mathbf{e}_4) &= 1 \\ \mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_4) &= -3a_2 \end{aligned}$$

$\underline{m = 5}$

$$\mathbf{e}_5^*(\mathrm{Ad}[v_t]_*\mathbf{e}_5) = 1$$

Thus we have the following.

Corollary 2. *Let $[w_t]$ and Q_t be as in Definition 6, with $w_t = e^t(z + a_2 z^2 + \cdots)$ and*

$$Q_t = \sum_{s=-5}^{5} c_s(t) e^{-st} \zeta^{-s}.$$

Then the following quantities are independent of t.

$$\begin{aligned} I_1 &= c_1 + (a_3 - a_2^2)c_3 + (2a_4 - 6a_3 a_2 + 4a_2^3)c_4 + (3a_5 - 12a_4 a_2 - 5a_3^2 + 28a_3 a_2^2 - 14a_2^4)c_5 \\ I_2 &= c_2 - a_2 c_3 + a_2^2 c_4 + (a_4 - 2a_3 a_2)c_5 \\ I_3 &= c_3 - 2a_2 c_4 + (-a_3 + 4a_2^2)c_5 \\ I_4 &= c_4 - 3a_2 c_5 \\ I_5 &= c_5 \end{aligned}$$

Proof. This follows immediately from Corollary 1 and the computations above. □

Remark 19. The finiteness of the sums above follows from the assumption that the coefficients of Q_t are zero for $s \geq 6$. This is the case if Q_t is generated by P_M from Schiffer's equation for a coefficient functional of degree six or less.

If the functional is of lower degree, then one recovers the invariants by setting some of the c_s to zero in the above expressions. For example, for a functional involving coefficients up to a_3, we have $c_s(t) = 0$ for $s > 2$, so the integrals of motion are (suppressing dependence on t)

$$\begin{aligned} I_1 &= c_1 \\ I_2 &= c_2 \end{aligned}$$

Thus c_1 and c_2 are constants for a third order functional. This special case appears in the literature in different guises (see [8] Section III.1.4 for a discussion).

If the functional is of higher finite degree it is always possible to compute the I_m by the procedure above; moreover, the first terms of the invariants obtained will be the same as those given above.

6 Notation key

- **G** the group of matrix representations of functions holomorphic and univalent in a neighbourhood of 0
- $\mathfrak{g}$ the Lie algebra of **G**

- $[F]$ matrix element of $\mathbf{G}$ corresponding to the function F
- $[F]^m_n$ the entry of $[F]$ in the mth row and nth column
- $\langle h \rangle$ matrix element of $\mathfrak{g}$ corresponding to the function h
- $\langle h \rangle^m_n$ the entry of $\langle h \rangle$ in the mth row and nth column
- $\mathrm{T}_{[g]}\mathbf{G}$ tangent space to $\mathbf{G}$ at $[g]$
- $\mathrm{T}^*_{[g]}\mathbf{G}$ cotangent space to $\mathbf{G}$ at $[g]$
- $\mathrm{L}[F]_* : \mathrm{T}_{[g]}\mathbf{G} \to \mathrm{T}_{[F][g]}\mathbf{G}$ derivative of left multiplication map; in a matrix group, this is also given by left multiplication
- $\mathrm{L}[F]^* : \mathrm{T}_{[g]}\mathbf{G} \to \mathrm{T}_{[F]^{-1}[g]}\mathbf{G}$ given as follows: for $\alpha \in \mathrm{T}_{[g]}\mathbf{G}$, and $v \in \mathbf{T}_{[F]^{-1}[g]}\mathbf{G}$, $(\mathrm{L}[F]^*\alpha)(v) \equiv \alpha(\mathrm{L}[F]_* v)$
- $\mathrm{R}[F]_* : \mathrm{T}_{[g]}\mathbf{G} \to \mathrm{T}_{[g][F]}\mathbf{G}$ derivative of the right multiplication map, also given by right multiplication in a matrix group
- $\mathrm{Ad}[F] : \mathbf{G} \to \mathbf{G}$ given by $[g] \mapsto [F][g][F^{-1}]$
- $\mathrm{Ad}[F]_* : \mathfrak{g} \to \mathfrak{g}$ is the derivative of the previous mapping at the identity; given by $\langle zh \rangle \mapsto [F] \langle zh \rangle [F^{-1}]$
- $[,]$ is the Lie bracket, given by $[\langle zh \rangle , \langle zp \rangle] = \langle zh \rangle \langle zp \rangle - \langle zp \rangle \langle zh \rangle$
- $\mathrm{ad} \langle zh \rangle : \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad} \langle zh \rangle (\langle zp \rangle) = [\langle zh \rangle , \langle zp \rangle]$
- $\mathrm{ad} \langle zh \rangle^* : \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $(\mathrm{ad} \langle zh \rangle^* \alpha)(\langle zp \rangle) = \alpha(\mathrm{ad} \langle zh \rangle (zp))$ for all $\langle zp \rangle \in \mathfrak{g}$

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