[5] 1. Evaluate $\lim_{x\to 0} \frac{\sin^{-1}(3x)}{x}$.

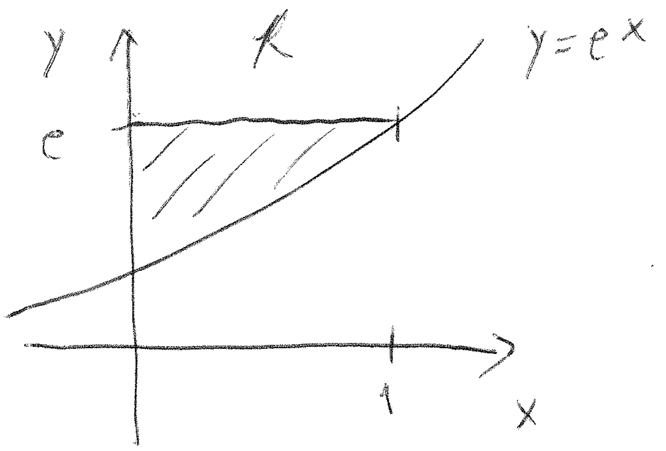
Try L'Hôpital's rule.

$$\stackrel{?}{=} \lim_{x\to 0} \frac{\frac{3}{\sqrt{1-(3x)^2}}}{1} = 3$$

Since this limit exists, by L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin^{-1}(3x)}{x} = 3.$$

[6] 2. Let $\mathcal{R}$ be the region in the first quadrant (that is, $x \geq 0$ and $y \geq 0$) between the curves $y = e$ and $y = e^x$. Using the cylindrical shell method, set up the integral for the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = 2$. **DO NOT EVALUATE THE INTEGRAL.**



$e = e^x \iff x = 1$

$$V = \int_0^1 2\pi(2-x)(e-e^x)\,dx$$

[5] 3. Evaluate $\int \frac{e^{2t}}{1+e^{4t}}\,dt$.

$$u = e^{2t}$$

$$du = 2e^{2t}\,dt \Rightarrow dt = \frac{du}{2u}$$

$$\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{u}{1+u^2}\,\frac{du}{2u} = \frac{1}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\tan^{-1}u + C$$

$$= \frac{1}{2}\tan^{-1}(e^{2t}) + C$$

[6] 4. Give the general form of the partial fraction decomposition of $\frac{x^3+2x+3}{(x-3)^2(x^2+x+1)^3}$. **DO NOT ATTEMPT** to find the values of the constants.

deg 3

$$\frac{x^3+2x+3}{(x-3)^2(x^2+x+1)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{(x^2+x+1)} + \frac{Ex+F}{(x^2+x+1)^2} + \frac{Gx+H}{(x^2+x+1)^3}$$

deg 8 > 3

<u>note</u> $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{1}{4} > 0$ has no factors

$\left[\text{or, } b^2-4ac = 1^2 - 4\cdot 1\cdot 1 = -3 < 0\right]$

[10] 5. Evaluate $\int_0^2 \frac{x+3}{x^2+6x+8}dx$.

$x^2+6x+8=(x+4)(x+2)$ so

$$\frac{x+3}{x^2+6x+8} = \frac{A}{x+4} + \frac{B}{x+2}$$

$\Rightarrow \quad x+3 = A(x+2) + B(x+4)$

set $x=-2 \Rightarrow \quad 1 = B \cdot 2 \Rightarrow \boxed{B=\frac{1}{2}}$

set $x=-4 \Rightarrow \quad -1 = A(-2) \Rightarrow \boxed{A=\frac{1}{2}}$

so $$\int_0^2 \frac{x+3}{x^2+6x+8}\,dx = \int_0^2 \left[\frac{1}{2}\frac{1}{x+2} + \frac{1}{2}\frac{1}{x+4}\right]dx$$

$$= \frac{1}{2}\ln|x+2| + \frac{1}{2}\ln|x+4| \Big|_0^2$$

$$= \frac{1}{2}\ln 4 + \frac{1}{2}\ln 6 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln 4 = \ln\sqrt{3} \quad \left(= \frac{1}{2}\ln 3\right)$$

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or use substitution

$u = x^2+6x+8$

$du = (2x+6)\,dx$

[5] 6. (a) Evaluate the improper integral $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$.

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx := \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{2-x}}\,dx$$

$$= \lim_{t\to 2^-} \Big|_0^t -2\sqrt{2-x}$$

$$= \lim_{t\to 2^-} \left[-2\sqrt{2-t} + 2\sqrt{2}\right] = 2\sqrt{2}.$$

[5] (b) Use the Comparison Theorem to determine whether the integral $\int_1^\infty \frac{1+\sin^2 x}{x^{1/3}}\,dx$ is convergent or divergent. **DO NOT** attempt to evaluate the integral directly.

$\cancel{\frac{1}{x^{1/3}} \quad \frac{1+\sin^2 x}{x^{1/3}}}$

Since $x > 0$, $\frac{1}{x^{1/3}} > 0$. Since $-1 \le \sin x \le 1$,

$0 \le \sin^2 x \le 1$ so $1 \le 1 + \sin^2 x$.

So $0 \le \frac{1}{x^{1/3}} \le \frac{1+\sin^2 x}{x^{1/3}}$

Since $\int_1^\infty \frac{1}{x^{1/3}}\,dx$ diverges by the p-test

$(\frac{1}{3} \le 1)$, $\int_1^\infty \frac{1+\sin^2 x}{x^{1/3}}\,dx$ diverges by

the comparison test.

[10] 7. Find the arc length of the graph of the function $f(x) = \ln(\sec(x))$ on the interval $0 \le x \le \frac{\pi}{4}$.

$$\text{length} = \int_0^{\pi/4} \sqrt{1 + f'(x)^2}\, dx \qquad f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx \qquad (1 + \tan^2 x = \sec^2 x)$$

$$= \int_0^{\pi/4} \sec x \, dx \qquad \sec x \ge 0 \text{ on } [0, \tfrac{\pi}{4}]$$

$$\text{so } |\sec x| = \sec x = \sqrt{\sec^2 x}$$

$$= \Big|_0^{\pi/4} \ln|\sec x + \tan x|$$

$$= \ln|\sec(\tfrac{\pi}{4}) + \tan(\tfrac{\pi}{4})| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln(1 + 0) = \ln(\sqrt{2} + 1)$$

[5] 8. Consider the graph of $f(x) = 1/x$ on the interval $[1, 5]$. Write down an integral that represents the surface area for the surface formed by rotating the graph around the x-axis. **DO NOT EVALUATE** the integral.

$$\text{Surface Area} = \int_1^5 2\pi y \sqrt{1 + f'(x)^2}\, dx$$

$$f'(x) = \frac{-1}{x^2} \qquad = \int_1^5 2\pi \left(\frac{1}{x}\right) \cdot \sqrt{1 + \frac{1}{x^4}}\, dx$$

$$\Rightarrow f'(x)^2 = \frac{1}{x^4}$$

[5] 9. (a) Find the area bounded by the spiral $r = \theta$, $0 \leq \theta \leq \pi/2$ (given here in polar coordinates) and the y-axis.

$$\text{Area} = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2} \theta^2 \, d\theta$$

$$= \Big|_0^{\pi/2} \frac{\theta^3}{6} = \frac{\pi^3}{48}$$

[7] (b) Find the area bounded by the parametric curve $x = t^3$, $y = 2t^3 + 1$, $0 \leq t \leq 1$, and the x-axis.

$$\text{Area} = \int_0^1 y \, dx = \int_0^1 (2t^3 + 1)(3t^2) \, dt$$

$$= \int_0^1 (6t^5 + 3t^2) \, dt$$

$$= \cancel{\Big|_0^1 (t^6 + t^3) = 2}$$

$$= \Big|_0^1 t^6 + t^3 = 2$$

10. Consider the parametric curve $x = t + 2\ln t$, $y = t - 3\ln t$ for $t > 0$.

[3] (a) Find the slope $\dfrac{dy}{dx}$ of the curve as a function of t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - \frac{3}{t}}{1 + \frac{2}{t}} = \frac{t-3}{t+2}$$

[5] (b) Find $\dfrac{d^2y}{dx^2}$ as a function of t.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{t-3}{t+2}\right)}{1+\frac{2}{t}}$$

$$= \frac{(t+2)-(t-3)}{(t+2)^2} \cdot \frac{1}{1+\frac{2}{t}}$$

$$= \frac{5t}{(t+2)^3}$$

[3] (c) On what interval is the curve concave up? On what interval is it concave down? (Note: we are only considering $t > 0$.)

Since $\dfrac{d^2y}{dx^2} = \dfrac{5t}{(t+2)^3} > 0$ on $(0, \infty)$,

the curve is concave up on $(0, \infty)$

and concave down nowhere.

11. Consider the polar curve $r = 1 - \sin\theta$.

[4] (a) Find the Cartesian coordinates of the point on the curve such that $\theta = \pi/4$.

$$x = r\cos\theta = (1-\sin\theta)\cos\theta$$

at ~~the~~ $\theta = \pi/4$, $x = (1-\sin(\tfrac{\pi}{4}))(\cos\tfrac{\pi}{4}) = (1-\tfrac{1}{\sqrt{2}})\cdot\tfrac{1}{\sqrt{2}}$

$$y = r\sin\theta = (1-\sin\theta)\sin\theta$$

at $\theta = \pi/4$

$$y = (1-\sin(\tfrac{\pi}{4}))\sin(\tfrac{\pi}{4}) = (1-\tfrac{1}{\sqrt{2}})\tfrac{1}{\sqrt{2}}.$$

[6] (b) Find the slope of the polar curve at $\theta = \pi/4$.

$r = 1-\sin\theta$

$\frac{dr}{d\theta} = -\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta - 2\cos\theta\sin\theta}{\sin^2\theta - \cos^2\theta - 1}$$

So
$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\cos(\frac{\pi}{4}) - 2\cos(\frac{\pi}{4})\sin(\frac{\pi}{4})}{\sin^2(\frac{\pi}{4}) - \cos^2(\frac{\pi}{4}) - 1}$$

$$= \frac{\frac{1}{\sqrt{2}} - 1}{-1} = 1 - \frac{1}{\sqrt{2}}.$$