Problems for MATH 4710 *This is not a problem set to be handed in!*

Note: There are a few exercises which I answered or will answer in class; I included them here because they belonged with other similar exercises.

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Branch cuts

1. Show that the function $\log_0 z$ defined on $\mathbb{C}\backslash L_0$ cannot be continuously extended to $\mathbb{C}\backslash\{0\}$.

2. Write expressions for the two branches of square root on $\mathbb{C}\backslash L_{-\pi}$. That is, if $z = re^{i\psi}$ for $\psi \in (-\pi, \pi)$, given expressions for $\sqrt{z}$ in terms of r and ψ. Sketch the domain and range.

3. Write expressions for the three branches of cube root on $\mathbb{C}\backslash L_0$. Sketch the domain and range.

Riemann surfaces

4. (a) Verify that the Riemann sphere is a Riemann surface with charts $\phi(z) = 1/z$ on $\mathbb{C}_\infty\backslash\{0\}$ and $\psi(z) = z$ on $\mathbb{C}$.

   (b) Show that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a holomorphic map of Riemann surfaces (in the special sense given in the second lecture) if and only if f is meromorphic and not identically equal to ∞.

5. Let $\Gamma = \{m\tau_1 + n\tau_2 : m, n \in \mathbb{Z}\}$ where τ_1 and τ_2 are real linearly independent complex numbers. Let $R = \mathbb{C}/\Gamma$ be the torus as defined in class.

   (a) Show that any holomorphic function $f : R \to \mathbb{C}$ is constant.

   (b) Let $f : R \to \mathbb{C}_\infty$ be a meromorphic function (equivalently, a holomorphic map between the Riemann surfaces R and $\mathbb{C}_\infty$ which is not identically equal to ∞). Show that the sum of the residues of f is zero. *Hint*: let Γ be the curve consisting of four sides of a parallelogram which is a translation of the parallelogram spanned by τ_1 and τ_2, traced with positive orientation. For some translation there are no poles on Γ. Why is the integral zero?

   (c) Assume that f is not constant. Show that the number of zeros equals the number of poles, by considering the integral
   $$\int_\Gamma \frac{f'(z)}{f(z)} dz$$
   and applying the argument principle.

   (d) Modify the previous exercise to show that if f is not constant, then f takes on each value in $\mathbb{C}_\infty$ the same number of times counting multiplicity. *Note*: this is a generic property of meromorphic functions on compact Riemann surfaces.

Topology and geometry of the $\mathbb{C}_\infty$, $\mathbb{C}$ and $\mathbb{D}$

6. Let f be holomorphic on a domain $U \subset \mathbb{C}_\infty$. Show that f is a continuous map from $(\mathbb{C}_\infty, d_s)$ into $(\mathbb{C}, d_e)$.

7. Let f be meromorphic on a domain $U \subset \mathbb{C}$. Show that f is a continuous map from U into $\mathbb{C}_\infty$. (It's enough to show that f is continuous at each pole).

8. Let f be meromorphic on a domain $U \subset \mathbb{C}_\infty$. Show that f is a continuous map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. (By the previous exercise, it's enough to show that f is continuous at ∞, if $\infty \in U$.)

9. (a) For $k > 0$, $k \in \mathbb{Z}$, show that $f(z) = z^{-k}$ for $k < 0$, is holomorphic on $\mathbb{C}_\infty \backslash \{0\}$.
   (b) For $k > 0$, $k \in \mathbb{Z}$, show that $f(z) = z^k$ is meromorphic on $\mathbb{C}_\infty$ (with a pole of order k at ∞).

10. Prove the formula for stereographic projection using trigonometry: for $z = x + iy$ in the plane, the point (x_1, x_2, x_3) on the surface is given by

$$\begin{aligned} x_1 &= \frac{2x}{|z|^2+1} \\ x_2 &= \frac{2y}{|z|^2+1} \\ x_3 &= \frac{|z|^2-1}{|z|^2+1}. \end{aligned}$$

and

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

11. This exercise shows that the formula for the spherical length of a curve that I gave in class, really is the length along the sphere. Let $z = x + iy \in \mathbb{C}$, and (x_1, x_2, x_3) be the corresponding point on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Let ϕ and θ denote the spherical coordinates of this point

$$\begin{aligned} x_1 &= \sin\phi\cos\theta \\ x_2 &= \sin\phi\sin\theta \\ x_3 &= \cos\phi. \end{aligned}$$

Finally, let $(\phi(t), \theta(t))$, $a \le t \le b$ be the spherical coordinates of a curve on the sphere, and

$$\alpha(t) = (\, x_1(\phi(t), \theta(t))\, ,\, x_2(\phi(t), \theta(t))\, ,\, x_3(\phi(t), \theta(t))\,)$$

be the same curve in Cartesian coordinates.

   (a) Show that the length of α is

$$\int_a^b \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}\,dt = \int_a^b \sqrt{\dot{\phi}^2 + \sin^2\phi\,\dot{\theta}^2}\,dt.$$

   (b) Show that

$$z = \frac{\sin\phi}{1 - \cos\phi} e^{i\theta}.$$

   Of course you'll need the formula for stereographic projection.

   (c) Show that, if $z = \gamma(t)$ is the curve traced in the plane under stereographic projection, that

$$\int_\gamma \frac{2|dz|}{1 + |z|^2} = \int_a^b \sqrt{\dot{\phi}^2 + \sin^2\phi\,\dot{\theta}^2}\,dt.$$

12. (a) Show that the map $T(z) = e^{i\theta}/z$ satisfies

$$\frac{|T'(z)|}{1 + |T(z)|^2} = \frac{1}{1 + |z|^2}.$$

   (b) Let $\gamma : [a, b] \to \mathbb{C}_\infty \backslash \{0\}$ be a curve which passes through ∞ at a single point $c \in (a, b)$ and is smooth. (That is, $\gamma(t)$ is C^∞ on $(a, c) \cup (c, b)$ and there is an open interval I containing c such that $1/\gamma(t)$ is smooth.) Use part (a) to show that

$$\int_\gamma \frac{|dz|}{1 + |z|^2} = \int_{1/\gamma} \frac{|dz|}{1 + |z|^2}.$$

   Note: this allows us to define the length of curves through ∞.

13. (a) Show that the chordal and spherical distances are related by

$$d_c(z, w) = 2\sin\left(d_s(z, w)/2\right).$$

Use a trigonometric argument and the fact that under stereographic projection, the spherical distance is the length of a great circle on the unit sphere, while the chordal distance is the distance in $\mathbb{R}^3$. *Hint*: the spherical distance equals the angle subtended by the rays joining $(0, 0, 0)$ to each of the two points on the sphere.

(b) Conclude that for any $z \in \mathbb{C}_\infty$ and $r \in [0, \pi]$, $B_c(z, 2\sin(r/2)) = B_s(z, r)$. *Note*: I made a mistake in class on page 3 of Lecture 5. This corrects it. (Either the domain of f has to be changed on page (2) or the equation relating B_c and B_s should be changed).

14. Prove that if (x_1, x_2, x_3) on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ maps to the point z in the plane under stereographic projection based at the north pole $(0, 0, 1)$, then it maps to $w = 1/\bar{z}$ under stereographic projection based at the south pole $(0, 0, -1)$. *Hint*: try working with a great circle through the north and south pole to determine how $|w|$ and $|z|$ are related; the relation between their arguments is easy to find.

15. (a) Show that $d_s(\bar{z}, \bar{w}) = d_s(z, w)$.
(b) Show that $d_s(-z, -w) = d_s(z, w)$.
(c) Show that $d_s(1/z, 1/w) = d_s(z, w)$.
(d) Show that $d_s(-1/\bar{z}, -1/\bar{w}) = d_s(z, w)$.

16. Show that the points z and $-1/\bar{z}$ are antipodal; i.e. they lie on opposite sides of the sphere.

17. (a) Show that if

$$T(z) = e^{i\theta}\frac{(z-a)}{1+\bar{a}z}, \quad a \in \mathbb{C}$$

then

$$\frac{|T'(z)|^2}{(1+|T(z)|^2)^2} = \frac{1}{(1+|z|^2)^2}.$$

(b) Show that if

$$T(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \quad a \in \mathbb{D}$$

then

$$\frac{|T'(z)|^2}{(1-|T(z)|^2)^2} = \frac{1}{(1-|z|^2)^2}.$$

18. (a) Define

$$\mathrm{Isom}(d_s) = \left\{T \,:\, T(z) = e^{i\theta}\frac{z-a}{1+\bar{a}z}\right\} \cup \left\{T \,:\, T(z) = -\frac{e^{i\theta}}{z}\right\}.$$

Show that $\mathrm{Isom}(d_s)$ forms a group under composition.

(b) Define

$$\mathrm{Aut}(\mathbb{D}) = \left\{T \,:\, T(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \quad a \in \mathbb{D}\right\}.$$

Show that $\mathrm{Aut}(\mathbb{D})$ forms a group under composition.

19. Show that the set of one-to-one, onto analytic maps from $\mathbb{D}$ to $\mathbb{D}$ is

$$\left\{T(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \,:\, a \in \mathbb{D},\ \theta \in [0, 2\pi)\right\}.$$

To do this proceed as follows:

(a) Show that the maps T of the above form are one-to-one onto maps of the disc (this follows easily from properties of Möbius transformations, if you first verify that they take the unit circle $|z| = 1$ to itself).

(b) If $f : \mathbb{D} \to \mathbb{D}$ is one such map, there's an $a \in \mathbb{D}$ such that $f(a) = 0$. Let

$$T(z) = \frac{z - a}{1 - \bar{a}z}.$$

Apply the Schwarz lemma to $T \circ f^{-1}$ and $f \circ T^{-1}$.

Möbius transformations, rational functions and conformal mapping

20. (a) Find a bijective holomorphic map taking the closure of $\mathbb{D} = \{z : |z| < 1\}$ to the closure of $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ such that -1 maps to -1, $-i$ maps to 0 and 1 maps to 1.

    (b) Find a bijective holomorphic map taking $\mathbb{D}$ onto $A = \{z : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

21. Show that every rational map $f(z) = P(z)/Q(z)$ in lowest terms (that is, so that P and Q have no common factors) is an n to 1 map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, where $n = \max\{\deg(P), \deg(Q)\}$. More precisely, show that for each $p \in \mathbb{C}_\infty$ there are precisely n solutions to $f(z) = p$ counting multiplicity.

22. In this exercise, you may find the immediately preceding exercise useful. *Hint*: what is $f(1/z)$?

    (a) What is the image of $\mathbb{D}$ under the map $f(z) = z + 1/z$?

    (b) What is the image of $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ under the map $f(z) = z + 1/z$?

    (c) Show that f maps circles $|z| = R$ (for $R \neq 1$) to ellipses $x^2/a + y^2/b = 1$. What are a and b in terms of R?

23. Let z_1, z_2 and z_3 be distinct points in $\mathbb{C}_\infty$. Show that there is a *unique* Möbius transformation taking z_1 to 1, z_2 to 0 and z_3 to ∞.

24. Denote two-by-two complex matrices by $M_{2\times 2}(\mathbb{C})$. Define

$$\mathrm{GL}(2, \mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) : \det A \neq 0\},$$

$$\mathrm{SL}(2, \mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) : \det A = 1\}$$

and

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}.$$

where the slash denotes the quotient of groups and I is the identity matrix. Let Möb denote the set of Möbius transformations.

    (a) Show that the map

$$\begin{aligned} \phi : \mathrm{GL}(2, \mathbb{C}) &\to \text{Möb} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \tfrac{az+b}{cz+d}. \end{aligned}$$

    is a group homomorphism.

    (b) Show that $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to the group $G = \mathrm{GL}(2, \mathbb{C})/\{\lambda I : \lambda \in \mathbb{C}\backslash\{0\}\}$.

    (c) Show that Möb and G are isomorphic as groups.

25. (a) Show that every Möbius transformation taking $\mathbb{H}$ to $\mathbb{H}$ bijectively can be represented as

$$T(z) = \frac{az + b}{cz + d} \quad \text{for } a, b, c, d \in \mathbb{R}.$$

(b) Show that the group of Möbius transformations taking $\mathbb{H}$ to $\mathbb{H}$ bijectively is isomorphic to

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}.$$

Hint: it is easier to show that it isomorphic to $\mathrm{GL}(2,\mathbb{R})/\{\lambda I : \lambda \in \mathbb{R}\backslash\{0\}\}$, and that this latter group is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

(c) Show that the group of Möbius transformations taking $\mathbb{D}$ to $\mathbb{D}$ bijectively is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

26. Let $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\} \subset \mathbb{C}_\infty$.

   (a) Show that $J(z) = 1/z$ is a one-to-one onto meromorphic map from $\mathbb{D}$ to $\mathbb{D}^*$.

   (b) Let $f : \mathbb{C} \to \mathbb{D}^*$ be meromorphic. Show that f is constant.

   (c) Show that if $f : \mathbb{C} \to D$ is meromorphic, and D is a half-plane or disc in $\mathbb{C}_\infty$, then f is constant.

27. Find a Möbius transformation T taking $\mathbb{D}$ onto $\mathbb{D}$, such that $T(0) = i/2$ and $T'(0) > 0$.

28. Find a one-to-one, onto conformal map f from $\mathbb{D}$ to $\mathbb{C}\backslash(-\infty, -1/4]$ satisfying $f(0) = 0$ and $f'(0) = 1$. *Hint*: if you can find a bijective conformal map onto $\mathbb{C}\backslash(-\infty, 0]$ which takes 0 onto 1/4, you're almost there.

29. Find a one-to-one, onto conformal map from $\mathbb{D}$ to $\{z : 0 < \arg(z) < \pi/3\}$.

30. (a) Show that if $f : G \to H$ is an analytic map between domains G and H, and u is harmonic on H, then $u \circ f$ is harmonic on G.

   (b) Find a harmonic function on $A = \{z : 1 < |z| < R\}$ for some $R > 1$ which extends continuously to the boundary, is 1 on $|z| = R$, and is 0 on $|z| = 1$.

   (c) Fix $R < 1$. Let B be the open set bounded on the outside by the ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

   where $a = R + 1/R$ and $b = 1/R - R$, and on the inside by the line segment $y = 0, -2 \leq y \leq 2$. Find a harmonic function on B which extends continuously to the closure of B, and which is 1 on the outside boundary and 0 on the inside boundary. *Hint*: use parts (a) and (b), and find a suitable map f somewhere in this problem set.

   (d) Is the continuous extension harmonic on the domain bounded by the outer ellipse? Why or why not?

Argument principle and consequences

31. Let f be meromorphic, and let p be a pole of f of order m. Show that the residue of f at p is $-m$.

32. Use Rouché's theorem to prove the fundamental theorem of algebra: if $p(z) = p_0 + p_1 z + \cdots p_n z^n$ with $p_n \neq 0$, then p has precisely n roots in the plane counting multiplicity. *Hint*:

$$\lim_{z\to\infty} \frac{p(z) - p_n z^n}{p_n z^n} = 0.$$

33. Use Rouché's theorem to find the number of zeros of $p(z) = z^6 - 5z^5 + z^2 - 1$ in $\mathbb{D}$. *Hint:* It's actually the z^5 term that dominates on the circle $|z| = 1$.

34. Let G be a domain, and γ a simple closed contour homotopic in G to a point. Let f and h be analytic on g. Assume f has no zeros on the curve γ, and has zeros a_i $i = 1, \dots, n$ enclosed by the curve. Let $m(a_i)$ be their multiplicities. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}h(z)\,dz = \sum_{i=1}^{n} h(a_i)m(a_i).$$

Hint: mimic the proof of the root-pole counting theorem. (If you wanted to, you could generalize this result to allow poles and that the curve is not simple).

35. Let $p(z) = p_0 + p_1 z + \cdots p_n z^n$ be a polynomial. Let γ be a circle, traced counter-clockwise, which is large enough that it encloses all of the roots. Show that

$$\frac{1}{2\pi i}\int_\gamma z\frac{p'(z)}{p(z)}\,dz = \sum_{k=1}^{q} m(z_k)z_k$$

where the sum is over each of the q roots z_k and $m(z_k)$ is the multiplicity of the root z_k.

36. Let G be a domain. Let f be analytic and one-to-one.
    (a) Prove that for any $z_0 \in G$, the image of f contains an open neighbourhood of $f(z_0)$.
    (b) Let $z_0 \in G$. Prove that there is a small circle γ centred on z_0, and a disc $B(f(z_0, r)$, so that

    $$f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z) - w}dz$$

    for all $w \in B(f(z_0, r))$. You will need problem 25. (This is called the Bürmann-Lagrange formula).

Topology of spaces of continuous and analytic maps, normal families

37. Prove Lemma 1.5 in Section 7.1.

38. Prove Lemma 2.8 in Section 7.2: $\mathcal{F} \subset H(G)$ is locally bounded if and only if $f(z)$ is bounded uniformly on each compact set (in the sense that $\sup |f(z)| \leq M$ for all $z \in K$ and $f \in \mathcal{F}$).

39. Let $\mathcal{F}$ be a family of holomorphic functions on an open connected set G, all of which satisfy the normalization $f(z_0) = c$ for some fixed $z_0 \in G$. Show that if $\{f' : f \in \mathcal{F}\}$ is locally bounded, then $\mathcal{F}$ is a normal family in $C(G, \mathbb{C})$.

40. The "Dirichlet integral" or "Dirichlet energy" of a holomorphic function is the integral

$$\|f\|^2 = \int_{\mathbb{D}} |f'|^2 < \infty.$$

    (a) Define $\mathcal{D}_M = \{f \in H(\mathbb{D}) : \|f\| \leq M\}$. Show that $\mathcal{D}_M$ is a normal family.
    (b) The "Dirichlet space" $\mathcal{D}$ is the set of analytic functions on $\mathbb{D}$ satisfying $f(0) = 0$ and $\|f\| < \infty$. Show that the Dirichlet space is *not* a normal family in $C(\mathbb{D}, \mathbb{C})$. *Hint*: construct a sequence in $\mathcal{D}$ without a convergent subsequence.
    (c) Show that the Dirichlet space is not a normal family in $C(\mathbb{D}, \mathbb{C}_\infty)$. *Hint*: find a sequence of functions in $\mathcal{D}$ such that the spherical derivative is not locally bounded.

41. Let $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \in H(\mathbb{D}),\ f \text{ one-to-one},\ f(0) = f'(0) - 1 = 0\}$. It can be shown that for any $f \in \mathcal{S}$ and $z \in \mathbb{D}$, the following "growth estimate" holds:

$$|f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

You may use this freely in the following question.

(a) Show that $\mathcal{S}$ is a normal family.

(b) Show that if $f_n \in \mathcal{S}$ for each n, and $f_n \to f$ uniformly on compact sets, then $f \in \mathcal{S}$. *Hint*: Show that f cannot be constant. Then assume $f(z_1) = f(z_2) = \alpha$ for distinct points $z_1, z_2 \in \mathbb{D}$ and use Hurwitz' theorem to obtain a contradiction.

(c) Show that $\mathcal{S}$ is compact in $C(\mathbb{D}, \mathbb{C})$.

(d) Let the functional $a_n : H(\mathbb{D}) \to \mathbb{C}$ be defined by

$$a_n(f) = \frac{1}{n!} \frac{f^n(0)}{f'(0)}$$

(that is $a_n(f)$ is the nth coefficient in the Taylor series). Show that

$$\sup_{f \in \mathcal{S}} |a_n(f)| < \infty^\dagger.$$

Hint: show that the functional a_n is continuous.

The correct upper bound $|a_n| \leq n$ was conjectured* by L. Bieberbach in 1916, and not proven until 1984 by Louis deBranges. Unlike other famous conjectures, almost nothing of interest hinges on whether or not it is true. On the other hand, like many famous conjectures, the ideas developed to make progress on the problem are now fundamental in many fields, such as Teichmüller theory, hyperbolic geometry, complex dynamics, conformal field theory, and stochastic processes. By the way, the function taking on the upper bound $|a_n| = n$ is somewhere on this problem set.

*If you could call it a conjecture: all Bieberbach said was "Vielleicht ist überhaupt $k_n = n$" which translates as "maybe the upper bound is $k_n = n$" - not exactly a confident proclamation. The guess was apparently based only on the fact that $|a_2| \leq 2$. Talk about luck.

† One final remark: the usual derivation of the growth estimate starts with the estimate $|a_2| \leq 2$, so one might claim that part (c) is circular (or at least that one must establish the estimate $|a_2| \leq 2$ holds before going further). However it is a little-known fact that one can prove the growth estimate without the estimate on a_2.

42. (a) Let g be a holomorphic function on $\mathbb{D}$ and let f be holomorphic and one-to-one on $\mathbb{D}$. Assume that $g(0) = f(0) = 0$. Let $\mathbb{D}_r = \{z : |z| < r\}$. Show that if $g(\mathbb{D}) \subseteq f(\mathbb{D})$ then $g(\mathbb{D}_r) \subset f(\mathbb{D}_r)$ for all $0 < r < 1$. (This is called the "principle of subordination"). *Hint*: Use the standard form of the Schwarz lemma.

Note that this immediately implies that $g(\mathrm{cl}\mathbb{D}_r) \subseteq f(\mathrm{cl}\mathbb{D}_r)$.

(b) Let $\mathcal{P} = \{p \in H(\mathbb{D}) : p(0) = 1 \text{ and } \mathrm{Re}(p(z)) > 0\}$. Show that for $p \in \mathcal{P}$, whenever $|z| \leq r$

$$|p(z)| \leq \frac{1+r}{1-r}.$$

(In fact, this shows that $|p(z)| \leq (1+|z|)/(1-|z|)$ for all $z \in \mathbb{D}$.) *Hint*: set $f(z) = (1+z)/(1-z)$ in the previous exercise.

(c) Show that $\mathcal{P}$ is a normal family.

43. Let $\mathcal{F} = \{f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f''(z)| \leq M \text{ and } f(0) = f'(0) - 1 = 0\}$. Show that $\mathcal{F}$ is normal.

44. Let $p(z)$ be a polynomial of degree n. Let

$$\mathcal{F} = \{f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f^{(n+1)}(z)| \leq M \;\text{ and }\; f - p \;\text{ has a zero of order n at 0}\}.$$

Show that $\mathcal{F}$ is a normal family.

45. Invent and prove a weaker criterion for normality, in the previous two exercises.

46. The "Bloch norm" of a function f is

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|.$$

(a) Let $T : \mathbb{D} \to \mathbb{D}$ be a one-to-one onto analytic map. Show that $(1-|z|^2)|T'(z)| = 1-|T(z)|^2$.

(b) Let

$$\mathcal{F}_M = \{f \in H(\mathbb{D}) \,:\, \|f\|_{\mathcal{B}} \le M, \quad \text{and} \quad f(0)=0\}.$$

Show that if $f \in \mathcal{F}_M$ then $g(z) = f\circ T(z) - f\circ T(0) \in \mathcal{F}_M$.

(c) Show that $\mathcal{F}_M$ is a normal family.

47. The "Bloch space" is

$$\mathcal{B} = \{f \in H(\mathbb{D}) \,:\, \|f\|_{\mathcal{B}} < \infty\}$$

where the norm $\|\cdot\|_{\mathcal{B}}$ was given in the previous exercise.

(a) Show that $\mathcal{B}$ is not a normal family. (This should be very easy, if you apply the right theorem).

(b) Let

$$\mathcal{B}' = \{f \in \mathcal{B} \,:\, f(0)=0\}.$$

Show that $\mathcal{B}'$ is not normal either.

(c) Let

$$\mathcal{B}_n = \{f \in \mathcal{B} \,:\, f^{(k)}(0)=0,\ k=1,\dots,n\}.$$

Show that $\mathcal{B}_n$ is not normal.

After the previous exercise, it should be plausible that *no* collection of normalizations will shrink $\mathcal{B}$ to a normal family.

48. (a) Show that every hyperbolic disc $B_h(a,r) = \{z : d_h(a,z) < r\}$ for $a \in \mathbb{D}$, $r > 0$, is contained in a Euclidean disc $B(0,R)$ with $R < 1$. You may use the triangle inequality for the hyperbolic (Poincaré) metric.

(b) Show that every Euclidean disc $B(a,r)$ such that $\overline{B(a,r)} \subset \mathbb{D}$ is contained in some hyperbolic disc $B_h(0,R)$ for $R < \infty$.

49. In this exercise and the next one, keep in mind that the definition of equicontinuity of a family (from Conway and my lectures) depends on the metric of the space your functions map into.

(a) Recall that

$$\mu_{e,s}(f)(z) = \frac{2|f'(z)|}{1+|f(z)|^2}.$$

Show that for a domain $G \subset \mathbb{C}$ the family

$$\mathcal{F}_M = \{f : G \to \mathbb{C}_\infty \,:\, \mu_{e,s}(f)(z) \le M \quad z \in G\}$$

is equicontinuous in $C(G, \mathbb{C}_\infty)$. Don't use Theorem 3.8.

(b) Recall that

$$\mu_{h,e}(f)(z) = (1-|z|^2)|f'(z)|.$$

Show that the family

$$\mathcal{B}_M = \{f : \mathbb{D} \to \mathbb{C} \,:\, \mu_{h,e}(f)(z) \le M \quad z \in \mathbb{D}\}$$

is equicontinuous in $C(G, \mathbb{C})$.

(You might be tempted to use Arzela-Ascoli thoerem. This doesn't work because the family is not locally bounded, since $f \in \mathcal{B}_M \Rightarrow f + c \in \mathcal{B}_M$ for any $c \in \mathbb{C}$.)

50. In class, I defined new derivatives, depending on the metric. One of these was the Euclidean-to-spherical derivative

$$\mu_{e,s}(f)(z) = \frac{2|f'(z)|}{1+|z|^2}$$

and another was the hyperbolic-to-Euclidean derivative

$$\mu_{h,e}(f)(z) = (1-|z|^2)|f'(z)|.$$

Now we define a new derivative, hyperbolic to spherical:

$$\mu_{h,s}(f)(z) = \frac{2(1-|z|^2)|f'(z)|}{1+|f(z)|^2}.$$

(a) Show that

$$\lim_{w\to z} \frac{d_s(f(z), f(w))}{d_h(z,w)} = \mu_{h,s}(f)(z),$$

(b) Show that the family

$$\mathcal{F}_M = \{f : \mathbb{D} \to \mathbb{C}_\infty \;:\; \mu_{h,s}(f)(z) \leq M \quad \forall z \in \mathbb{D}\}$$

is normal.

(c) Let $\mathcal{F}$ be a family of meromorphic functions on $\mathbb{D}$. Show that if $\mu_{h,s}(f)$ is locally bounded, then $\mathcal{F}$ is normal in $C(\mathbb{D}, C_\infty)$.

51. Find the exercise on this list of problems that is equivalent to Proposition 3.3 section 7.3 in Conway.

(a) Observe that you have in fact proven this Proposition (provided you did the exercise).

(b) Observe that you have also proven a version of Proposition 3.3 with chordal distance replaced by spherical distance.

Harmonic functions and the Dirichlet problem

52. Let $A(r,R) = \{z : r < |z| < R\}$. Find the general solution u to the Dirichlet problem, with $u = \beta$ on the outer boundary and $u = \alpha$ on the inner boundary (α and β are constants).

53. In the previous exercise, set $R = 1$, $\alpha = 1$ and $\beta = 0$. For fixed z, what happens as $r \to 0$? Try guessing the limit before you try to find it.

54. Let f be a continuous function on $\partial\mathbb{D}$, and

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{\pi}^{\pi} P_r(\theta - t) f(e^{it})dt$$

be the solution to the Dirichlet problem with boundary values f. For α fixed define $\tilde{f}$ by $\tilde{f}(e^{it}) = f(e^{i(t+\alpha)})$ and let $\tilde{u}(re^{i\theta})$ be the corresponding solution to the Dirichlet problem with boundary values $\tilde{f}$. Show that $\tilde{u}(re^{i\theta}) = u(re^{i(\theta+\alpha)})$.

55. Conway Exercises X.1.1, X.1.2, X.1.4–X.1.7, X.1.9.

56. Conway Exercises X.2.1, X.2.2 (*Hint:* just observe that in the proof of Theorem 2.4, only the continuity of f at $e^{i\alpha}$ is used to show that $\lim_{z\to e^{i\alpha}} u(z) = f(e^{i\alpha})$), X.2.4.

57. X.2.5 *Hint*: you can use removability of singularities for holomorphic mappings, but keep in mind that the harmonic conjugate might have non-zero periods and thus the resulting holomorphic function would be multi-valued. If you are a bit sneaky, you can rig up an appropriate single-valued function.

Another approach is the following.

(a) Let G be an open connected domain and $f : \partial_\infty G \to \mathbb{R}$ be a continuous function. Show that if there is a solution u to the Dirichlet problem for f, then u is the Perron function.

(b) Consider the disk $B(a;r) \subset \mathbb{C}$ and let $B_0 = B(a;r)\backslash\{a\}$. Show that there is no harmonic function on G_0 with a continuous extension to $\mathrm{cl}\, B(a;r)$ which is zero on $\partial B(a;r)$ and which is non-zero at a.

Hint: if you could do this, you could certainly find a harmonic function which is one at a and zero on $\partial B(a;r)$. This contradicts something proven at the very end of section X.3.

(c) Problem X.2.5.

Hint: If $\mathrm{cl}\, B(a;r) \subset G$ you can solve the Dirichlet problem on $B(a;r)$ with boundary values equal to u on $\partial B(a;r)$ to obtain a (potentially) new harmonic function v. Apply part (b).

58. X.3.1 (a),(b),(c), X.3.2(a), X.3.4 (*Hint*: just observe that the continuity of f is never used in the proof; only the fact that f is bounded).

59. X.5 1(a),(b)(i)(ii), 3.

60. This exercise completes the proof of the special case of the Riemann mapping theorem that I gave in class. Let G be a simply connected, bounded domain in $\mathbb{C}$, whose boundary is a Jordan curve. Let $g(z,a)$ be Green's function of G with singularity at a. (Recall that a Jordan curve is a simple closed continuous curve in $\mathbb{C}$). Let $u(z) = g(z,a) + \log|z-a|$ and v be the complex conjugate of u. Let γ be a simple closed curve in G, which winds once around a.

    (a) Show that
    $$\int_\gamma \frac{\partial u}{\partial n} ds = 0.$$
    Hint: Use one of the special cases of Green's identity.

    (b) Use exercise 5 in "mapping theorems" to show that the change in v around γ is zero.

61. Show that the following domains are Dirichlet regions.

    (a) $\mathbb{D}\backslash[-1/2, 1/2]$.

    (b) Any domain bounded by n piecewise C^1 curves, with a smooth parametrization (i.e. the derivative does not vanish).

62. Give an example of a domain which is not a Dirichlet region.

63. Let G be a bounded, connected domain. Let h be continuous on ∂G. Let $\tilde{h}$ be the Perron function.

    (a) Show that if u solves the Dirichlet problem with boundary values h, then u is in the Perron family $\mathcal{P}(h, G)$. Conclude that $u \leq \tilde{h}$.

    (b) Show that $\tilde{h} \leq u$. Conclude that if u solves the Dirichlet problem with boundary values h, then u is the Perron function.

    (c) Thus if G is a Dirichlet domain, then the Perron function solves the Dirichlet problem.

Mapping theorems

64. Conway, exercises VII.4.2, VII.4.6, VII.4.7, VII.4.9.

65. Let $f(x+iy) = u(x,y)+iv(x,y)$ be an analytic function of $x+iy$. Show that the Jacobian determinant of the map $(x,y) \to (u,v)$ is
$$\frac{\partial(u,v)}{\partial(x,y)}(x,y) = |f'(x+iy)|^2.$$

66. Let $w(u, v)$ is a harmonic function on a domain D, and let $f(x+iy) = u(x, y) + iv(x, y)$ be an analytic function taking a domain G into D satisfying $f'(z) \neq 0$. Show that at any point $z = x + iy \in G$

$$|\nabla(w \circ f)(z)|^2 = |(\nabla w) \circ f(z)|^2 |f'(z)|^2.$$

67. Let $\gamma(t) = x(t) + iy(t)$ be a simple closed (positively oriented) smooth curve in $\mathbb{C}$. Let $n(t)$ denote the normal vector at $\gamma(t)$, pointing in the direction to the right of the direction of motion (that is, to the right of $\gamma'(t)$ where the prime denotes differentiation with respect to t). Let u be a C^1 function on the closed set whose boundary is γ. Let

$$\frac{\partial u}{\partial n}$$

denote the directional derivative of u in the direction of n. Show that

$$\frac{\partial u}{\partial n}|\gamma'(t)| = -\frac{\partial u}{\partial y}x' + \frac{\partial u}{\partial x}y'.$$

Thus if $ds = \sqrt{x'^2 + y'^2}dt = |\gamma'(t)|dt$ denotes infinitesimal arc length, we can say that

$$\frac{\partial u}{\partial n}ds = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Hint : The normal is γ' rotated by $\pi/2$ clockwise. Use this to show that

$$n(t) = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}}(y'(t), -x'(t)).$$

68. Let $f = u + iv$ be a holomorphic function on a domain G. Let γ be a smooth curve in G, with end-points z_0 and z_1 in that order. Show that

$$\int_\gamma \frac{\partial u}{\partial n}\, ds = v(z_1) - v(z_0).$$

Hint: Apply the Cauchy-Riemann equations.

69. Let G be a piecewise smoothly (C^2) bounded domain in $\mathbb{C}$ with four distinguished points z_0, z_1, z_2, z_3, whose ordering traces around ∂G counter-clockwise. Assume G is bounded by four smooth curves $C_1, \ldots, C_4$, where C_1 joins z_0 to z_1, C_2 joins z_1 to z_2, etc; and that $C_1 + C_2 + C_3 + C_4$ is positively oriented with respect to G. Assume that you can solve the following boundary value problem: there is a u such that u is harmonic on G, u has a C^2 extension to cl$G$, $u = 0$ on C_1, $u = 1$ on C_3, $\partial u/\partial n = 0$ on C_2 and C_4. (This means: if $t \mapsto x(t) + iy(t)$ is any smooth parametrization of C_2 or C_4, then

$$-\frac{\partial u}{\partial y}x' + \frac{\partial u}{\partial x}y' = 0.)$$

(a) Let v be the harmonic conjugate of u on G which is zero at z_1. Show that v is constant on C_2 and C_4, and $\partial v/\partial n = 0$ on C_1 and C_2. You may assume that u and v are C^1 up to the boundary. *Hint*: use the Cauchy-Riemann equations.

(b) Show that $f = u + iv$ is one-to-one. Conclude that there is a one-to-one analytic map from G onto a rectangle

$$R = \{u + iv \,:\, 0 < u < 1 \text{ and } 0 < v < r\}$$

for some $R > 0$, which takes z_0, z_1, z_2, z_3 onto $ri, 0, 1, 1 + ri$ in that order. *Hint*: Show that for any $w_0 \in R$, the change in argument of $f(z) - w_0$ around ∂G is 2π.

(c) Define the energy of the domain G with distinguished points $z_0, \ldots, z_3$ to be

$$E(G, z_0, \ldots, z_3) = \int\int_G |\nabla u|^2 \, dA$$

where u is the solution of the boundary value problem described above. Compute r in terms of $E(G)$. *Hint*: first compute

$$\int_{C_3} \frac{\partial u}{\partial n} ds.$$

After that, you'll need to apply Green's identity to the double integral and make use of the boundary values of u.

70. Let $R_r = \{x + iy : 0 < u < 1 \text{ and } 0 < v < r\}$, and let $z_0 = ir$, $z_1 = 0$, $z_2 = 1$, $z_3 = 1 + ir$. Let $C_1, \ldots, C_4$ be as in the previous question.

(a) What is the solution on R_r to the boundary value problem in the previous question? (Set $G = R_r$; the distinguished points and curves are described above).

(b) Find the energy $E(R_r, z_0, \ldots, z_3)$.

(c) Show that you can map any rectangle S with corners $w_0, \ldots, w_4$ onto R_r, so that w_i maps to z_i, with a one-to-one analytic map if

$$r = \frac{\text{length of side } w_3 w_2}{\text{length of side } w_2 w_1}.$$

(d) Show that if there is a one-to-one analytic map f from S onto R_r, then $E(S, w_0, \ldots, w_3) = E(R_r, z_0, \ldots, z_3)$. Hint: if u is the solution to the boundary value problem on R_r then $u \circ f$ is the solution to the boundary value problem on S. You may assume that f extends continuously to the boundary of S.

(e) Conclude that the condition in part (c) is in fact necessary for the existence of a one-to-one analytic onto map from S to R_r which takes z_i to w_i.