Polar Coordinates

MATH 1700

Readings

Section 10.3, 10.4

What are polar coordinates?

We can specify the location of a point (x, y) using its distance to the origin r and the angle θ between the ray through $(0, 0)$ and (x, y) and the x-axis.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

and

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

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Warning # 1: we allow r to be negative!
Warning # 2: there are many choices of (r, θ) for a given (x, y)!

Polar curves

An equation involving r and θ describes a set of points in the plane which satisfy that equation. This is called a **polar curve.** They are often written $r = f(\theta)$ (but not always).

Symmetries of polar curves

If a polar equation has an algebraic property, this can indicate a symmetry of the corresponding polar curve.

1. If the equation is unchanged when θ is replaced by $-\theta$, then the curve is symmetric about the polar axis (what would be the *x*-axis).
2. If the equation is unchanged when *r* is replaced by $-r$, then the curve is symmetric about the origin.
3. If the equation is unchanged when θ is replaced by $\pi - \theta$, then the curve is symmetric about the ray $\theta = \pi/2$ (what would be the *y*-axis).

Tangents to polar curves

If $r = f(\theta)$, then we can write

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

So the slope of the tangent is (note $dr/d\theta = f'(\theta)$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

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Special cases:
1. If $dx/d\theta = 0$ and $dy/d\theta \neq 0$ then the tangent is vertical
2. If $dy/d\theta = 0$ and $dx/d\theta \neq 0$ then the tangent is horizontal
3. At the origin $r = 0$, we have $dy/dx = \tan\theta$ so long as $dr/d\theta \neq 0$.

Area of a polar curve

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The area swept out by a polar curve between $\theta = a$ and $\theta = b$

$$A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2} f(\theta)^2 d\theta.$$

Area between two polar curves

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The area between these curves between $\theta = a$ and $\theta = b$ is

$$A = \int_a^b \frac{1}{2} \left(f(\theta)^2 - g(\theta)^2 \right) d\theta.$$

Arc length of a polar curve

Let $r = f(\theta)$, $a \leq \theta \leq b$ be a polar curve. This can be thought of as a parametric curve with $x = r\cos\theta$, $y = r\sin\theta$.

A bit of computation using the arc length formula for parametric curves gives us

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$