Integration of Rational Functions by Partial Fractions

MATH 1700

Readings

Readings: Section 7.4

Rational functions

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where P and Q are polynomials.

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If you need to find $\int \frac{P(x)}{Q(x)} dx$:

Step One: always make sure that the degree of the numerator is less than the degree of the denominator, by dividing to obtain:

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $S(x)$ is a polynomial.

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If $Q(x)$ is a product of distinct linear factors, then we have the partial fraction decomposition

$$\frac{R(x)}{(a_1x+b_1)\cdots(a_kx+b_k)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

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and this makes finding $\int \frac{R(x)}{Q(x)}dx$ easy, because

$$\int \frac{A}{ax+b} = \frac{A}{a}\ln|ax+b| + C.$$

Case II: $Q(x)$ is a product of linear factors, some repeated

If $Q(x) = (ax+b)^r$, then we use the partial fraction decomposition

$$\frac{R(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_r}{(ax+b)^r}. \tag{1}$$

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If $Q(x)$ has many such factors, we use one sum of the form (1) for each one. For example,

$$\begin{aligned}\frac{R(x)}{(ax+b)^r(cx+d)^s} = {} & \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_r}{(ax+b)^r} \\ & + \frac{B_1}{cx+d} + \frac{B_2}{(cx+d)^2} + \cdots \frac{B_s}{(cx+d)^s}\end{aligned}$$

Case III: $Q(x)$ has irreducible quadratic factors, none of which is repeated

If $Q(x)$ is a product of distinct quadratic factors, then we have the partial fraction decomposition

$$\frac{R(x)}{(a_1x^2+b_1x+c_1)\cdots(a_kx^2+b_kx+c_k)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_kx+B_k}{a_kx^2+b_kx+c_k}.$$

Finding the integral of the terms on the right hand side is a bit harder than the linear case (see next slide).

Case III: $Q(x)$ has irreducible quadratic factors, none of which is repeated

If $Q(x) = (ax^2 + bx + c)$ and this doesn't factor, this can be integrated using either the substitution

$u = ax^2 + bx + c$, $du = 2ax + b$,

and/or using

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

possibly after completing the square.

Mixed factors

If you have factors of Case I, II, or III (or IV, coming up), then you add the partial fractions decompositions corresponding to each term together. For example if $ax^2 + bx + c$ does not factor, then

$$\frac{R(x)}{(ax+b)^r(cx+d)(ex^2+fx+g)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_r}{(ax+b)^r} + \frac{B}{cx+d} + \frac{Cx+D}{ax^2+bx+c}.$$

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor

If $Q(x)$ has a factor of the form $(ax^2+bx+c)^r$, where ax^2+bx+c doesn't factor, then use the partial fractions decomposition

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Tips

Tips:

(1) Don't forget to simplify if possible.

(2) Look for an easy substitution. Is the denominator the derivative of the numerator (up to a constant multiple)?

(3) Make sure you cancel all common factors in the numerator and denominator. Have you missed any?

(4) Don't forget that quadratics $ax^2 + bx + c$ might factor into two linear factors.

Rationalizing substitutions

One last trick. (Phew!)

If you have a quotient and it isn't rational, sometimes a substitution of the form

$$u = \sqrt[n]{g(x)}$$

will turn it into one.