Curves defined parametrically

MATH 1700

Readings

Section 10.1, Section 10.2.

What is a parametric curve?

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As t varies over the interval, $(x, y) = (f(t), g(t))$ traces out a curve.

Three facts about parametric curves

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Fact #3: Parametric curves do not have to be graphs of functions.

Tangents

Assuming that $dx/dt = f'(t) \neq 0$, we can compute the slope

$$\text{slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Vertical tangent: occurs if $dx/dt = 0$ but $dy/dt \neq 0$.

Horizontal tangent: occurs if $dy/dt = 0$ but $dx/dt \neq 0$.

Convexity

If $dx/dt = f'(t) \neq 0$, then we can compute the derivative of the slope as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{dt}\frac{dy}{dx}\right] \Big/ \frac{dx}{dt}.$$

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The curve is concave up: when $d^2y/dx^2 > 0$
The curve is concave down: when $d^2y/dx^2 < 0$.

Area under a parametric curve

Let $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ be a parametric curve.
Assume $a = f(\alpha) < b = f(\beta)$.
The area under the curve and above the x-axis is

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt.$$

If on the other hand $a = f(\beta) < b = f(\alpha)$, then the area is given by

$$A = \int_a^b y dx = \int_\beta^\alpha g(t) f'(t) dt.$$

Definition of arc length of a parametric curve

Definition

Let $x = f(t)$, $y = g(t)$ be a parametric curve such that f' and g' are continuous on $[\alpha, \beta]$. Assume that the curve is traced exactly once as t increases from α to β. The length of the curve between α and β is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Why is this a good definition?

The length can be approximated by straight lines:

$$\begin{aligned} L &\cong \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= \sum_{i=1}^{n} \sqrt{f'(t_i^*)^2(\Delta t)^2 + g'(t_i^{**})^2(\Delta t)^2} \quad \text{(using MVT)} \\ &= \sum_{i=1}^{n} \sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2}\Delta t_i. \end{aligned}$$

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The actual length is the limit:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2}\Delta t_i = \int_{\alpha}^{\beta} \sqrt{f'(t)^2 + g'(t)^2}dt.$$

Surface area of a parametric curve

Let $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ be a parametric curve. Consider the surface obtained by rotating the curve around the x-axis.

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The surface area is

$$S = \int_{\alpha}^{\beta} 2\pi y \, ds$$
$$= \int_{\alpha}^{\beta} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$
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