# Improper integrals

MATH 1700

# Readings

**Readings**: Section 7.8

# Type I integrals: the interval is infinite

## Definition

1. Assume that $\int_a^t f(x)dx$ exists for all $t \geq a$. Then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided that this limit exists.

2. Assume that $\int_t^b f(x)dx$ exists for all $t \leq b$. Then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided that this limit exists.

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$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided that this limit exists.

**Terminology**: if the limit exists the integral is called **convergent** and if it does not exist it is called **divergent**.

# Type I integrals: the interval is doubly infinite

## Definition

Assume that $\int_a^\infty f(x)dx$ is convergent and $\int_{-\infty}^a f(x)dx$ is convergent. Then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx.$$

# The $p$-test for infinite intervals

**Theorem**

$$\int_1^{\infty} \frac{1}{x^p} dx$$

*converges if $p > 1$ and diverges if $p \leq 1$.*

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**Theorem**

$$\int_1^\infty \frac{1}{x^p}\,dx$$

*converges if $p > 1$ and diverges if $p \leq 1$.*

**Tip**: you should be able to prove this yourself.

# Type II integrals: discontinuous functions

## Definition

1. Let $f$ be continuous on $[a, b)$ and discontinuous at $b$. We define

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided the limit exists.

2. Let $f$ be continuous on $(a, b]$ and discontinuous at $a$. We define

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided the limit exists.

# Type II integrals: discontinuous functions

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1. Let $f$ be continuous on $[a, b)$ and discontinuous at $b$. We define

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided the limit exists.

2. Let $f$ be continuous on $(a, b]$ and discontinuous at $a$. We define

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided the limit exists.

**Terminology**: if the limit exists we say that the integral is **convergent**; otherwise it is **divergent**.

# Type II integrals: discontinuous functions continued

**Definition**

Let $f$ be continuous on $[a, b]$ except at the point $c \in (a, b)$. If $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

# The $p$ test for finite intervals

**Theorem**

$$\int_0^1 \frac{1}{x^p}\,dx$$

*converges if* $p < 1$ *and diverges if* $p \geq 1$*.*

# The $p$ test for finite intervals

**Theorem**

$$\int_0^1 \frac{1}{x^p}\,dx$$

*converges if $p < 1$ and diverges if $p \geq 1$.*

**Tip**: you should be able to prove this yourself.

# Comparison test

Sometimes you can determine whether an integral converges or diverges without actually evaluating it.

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### Theorem (Comparison Theorem)

*Let $f$ and $g$ be continuous functions on $[a, \infty)$ such that $f(x) \geq g(x) \geq 0$ for all $x \geq a$.*

1. *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.*

# Comparison test

Sometimes you can determine whether an integral converges or diverges without actually evaluating it.

**Theorem (Comparison Theorem)**

*Let $f$ and $g$ be continuous functions on $[a, \infty)$ such that $f(x) \geq g(x) \geq 0$ for all $x \geq a$.*

1. *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.*
2. *If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.*