Solving an inequality

Problem: Find the set of all x satisfying $|2x-4| \leq |3x|$.

Solution: Let $S = \{x : |2x-4| \leq |3x|\}$.

There are three cases, $x \leq 0$, $0 < x < 2$, and $2 \leq x$.

Case I: $x \leq 0$. In this case, $|3x| = -3x$, and $|2x-4| = 4-2x$ since $2x-4 \leq -4 < 0$. So (assuming throughout that $x \leq 0$)

$$|2x-4| \leq |3x| \Leftrightarrow 4-2x < -3x \Leftrightarrow x < -4.$$

So $x \leq 0$ and $x \in S$ if and only if $x < -4$.

Case II: $0 < x < 2$. In this case, $|3x| = 3x$, and $|2x-4| = 4-2x$ because $2x < 4$ so $2x-4 < 0$. So (assuming throughout that $0 < x < 2$)

$$|2x-4| \leq |3x| \Leftrightarrow 4-2x < 3x \Leftrightarrow 4 < 5x \Leftrightarrow x < 4/5.$$

So $0 < x < 2$ and $x \in S$ if and only if $4/5 < x < 2$.

Case III: $2 \leq x$. In this case, $|3x| = 3x$, and $|2x-4| = 2x-4$ because $2x \geq 4$ so $2x-4 \geq 0$. So (assuming throughout that $2 \leq x$)

$$|2x-4| \leq |3x| \Leftrightarrow 2x-4 < 3x \Leftrightarrow x > -4$$

which is always satisfied for $x \geq 2$. So $2 \geq x$ and $x \in S$ if and only if $x \geq 2$.

Since these are the only three cases, we have shown that $|2x-4| \leq |3x|$ if and only if $x > 4/5$ or $x < -4$. That is, $S = (-\infty, -4) \cup (4/5, \infty)$.